Homework #7. Due Thursday, October 22nd, in class

Reading:

1. For this homework assignment: contraction mapping theorem – class notes (Lectures 13-14); see also Rudin 9.3, pp. 220 and Kolmogorov-Fomin 8.1; parts of 4.3 and 4.4.

2. For next week's classes: Sections 7.2-7.5 in Rudin (uniform convergence, uniform convergence and continuity, uniform convergence and integration, uniform convergence and differentiation). Key theorems (we will definitely prove these in class): 7.8, 7.10, 7.12 (obtained as a consequence of 7.11), 7.18.

Problems:

1. (this is a generalization of Problem 1 on the first midterm). Let (X, d) be a metric space. Given a point $x \in X$ and a subset Z of X, we define $d(x, Z)$ (the distance from x to Z) by

$$d(x, Z) = \inf\{d(x, z) : z \in Z\}.$$

Note that one indeed has to take infimum, not minimum. There may be no $z \in Z$ which is closest to x.

(a) Prove that $d(x, Z) = 0 \iff x \in \overline{Z}$.

(b*) Prove that $d(x, Z) \geq d(y, Z) - d(x, y)$ for all $x, y \in X$ and $Z \subseteq X$

(c) Now fix $Z \subseteq X$, and define $d_Z : X \to \mathbb{R}$ by $d_Z(x) = d(x, Z)$. Use (b) to prove that d_Z is uniformly continuous.

Note: Problem 1 on the first midterm is the special case of (c) where Z is a single point.

2*. Again let (X, d) be a metric space. Let $a \in X$ and K a compact subset of X. Prove that there exists $k \in K$ such that $d(a, k) = d(a, K)$ (this is equivalent to saying that the set $\{d(a, z) : z \in K\}$ has the minimal element).

3. Let X be a metric space and Y a subset of X.

(a) Prove that if X is complete and Y is closed in X, then Y is complete.

(b) Prove that if Y is complete, then Y is closed in X.

Recall that we proved the analogous statements with 'complete' replaced by 'compact' (Lecture 6) and 'sequentially compact' (Homework 4).

4*. Recall the general setup of Newton's method. Let $I \subseteq \mathbb{R}$ be a closed interval, $f : I \to \mathbb{R}$ a differentiable function such that f'' is continuous on

I (the latter is a new condition we did not impose in class). Assume that $f'(x) > 0$ for all $x \in I$ and f changes sign on I, so that there exists unique $\alpha \in I$ such that $f(\alpha) = 0$.

Define the function $g : I \to \mathbb{R}$ by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

Given $x_0 \in I$, we will say that x_0 *is a valid starting value for Newton's method* if

(a) the sequence $\{x_n\}$ given by $x_n = g(x_{n-1})$ is defined for all n (for x_n to be defined we must have $x_{n-1} \in I$)

(b) the sequence $\{x_n\}$ defined in (a) converges to α (the unique root of f on I).

As discussed in class, if

(i) $g(I) \subseteq I$

(ii) there exists $C < 1$ such that $|g'(x)| \leq C$ for all $x \in I$,

then any $x_0 \in I$ is a valid starting value for Newton's method.

Now the actual problem: Show (using only assumptions on f in the first paragraph) that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

(1) $[\alpha - \delta, \alpha + \delta] \subseteq I$ and $g([\alpha - \delta, \alpha + \delta]) \subseteq [\alpha - \delta, \alpha + \delta]$

(2) $|g'(x)| \leq \varepsilon$ for all $x \in [\alpha - \delta, \alpha + \delta]$.

This implies that any x_0 sufficiently close to α is a valid starting value for Newton's method; moreover, we can make the contraction constant C as small as we want.

5. This problem describes a fancy way to show that closed bounded intervals in $\mathbb{R}$ are connected. We will say that a metric space (X, d) has the *chain property* if for any $x, y \in X$ and $\delta > 0$ there exists a finite sequence $x_0, x_1, \ldots, x_n$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_i, x_{i+1}) < \delta$ for all i.

(a)* Let X be a compact metric space with the chain property. Prove that X is connected.

(b) Prove that a closed bounded interval $[a, b] \subseteq \mathbb{R}$ has the chain property and deduce from (a) that $[a, b]$ is connected.

6*. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Assume that f' is bounded, that is, there exists $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous.

(b) Now assume that $f'(x) \to \infty$ as $x \to \infty$. Prove that f is not uniformly continuous.

7 (bonus). Prove that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is rational whenever x is irrational and vice versa $f(x)$ is irrational whenever x is rational.

Hint for 1(b): If S is a subset of $\mathbb{R}$ bounded below and $\alpha \in \mathbb{R}$, how to show that $\inf(S) \geq \alpha$?

Hint for 2: Apply Theorem 4.16 from Rudin to a suitable function.

Hint for 4: WOLOG we can assume that $\varepsilon < 1$. First show that there exists $\delta > 0$ satisfying (2). Then show that this δ also satisfies (1) (provided $\varepsilon < 1$) by estimating $|g(x) - g(\alpha)|$ for $x \in [\alpha - \delta, \alpha + \delta]$ from above.

Hint for 5(b): Assume that X is disconnected, and use Problem 2 from HW#6 and uniform continuity to reach a contradiction.

Hint for 6: Use the mean-value theorem.