Homework #6. Due Thursday, October 15th, in class

Reading:

1. For this homework assignment: Section 4.4 (continuity and connectedness) + class notes (Lectures 11-12).

2. For next week's classes: Fixed points and contraction mapping theorem (see Rudin 9.3, pp. 220 and Kolmogorov-Fomin 8.1), Sections 7.1-7.3 in Rudin (discussion of main problem, uniform convergence, uniform convergence and continuity).

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1*. (bonus) Problem 1(b) from HW#5.

2*. Let X be a metric space. Prove that X is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f(X) = \{1, -1\}$.

3.

(a*) Let X be a disconnected metric space, so that $X = A \sqcup B$ for some non-empty closed subsets A and B. Prove that if C is any connected subset of X, then $C \subseteq A$ or $C \subseteq B$.

(b*) A metric space X is called *path-connected* if for any $x, y \in X$ there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$ (informally, this means that any two points in X can be joined by a path in X). Prove that any path-connected metric space is connected.

4*. Let X be a metric space, $\{X_\alpha\}_{\alpha\in I}$ a collection (not necessarily finite) of subsets of X such that $\cap_{\alpha\in I} X_\alpha$ is non-empty and $\cup_{\alpha\in I} X_\alpha = X$. Prove that if each X_α is connected, then X is connected.

5. (practice) Metric spaces (X, d_X) and (Y, d_X) are called **isometric** if there exists a bijection $f : X \to Y$ such that $d_Y(f(a), f(b)) = d_X(a, b)$ for all $a, b \in X$. Prove that all abstract properties of metric spaces introduced in this class are preserved under isometries, that is, if (X, d_X) and (Y, d_Y) are isometric and X is compact, then Y is compact; if X is connected, then Y is connected etc.

6. Let (X, d_X) and (Y, d_Y) be metric spaces, and consider the product space $X \times Y$ with metric d given by $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ (this is indeed a metric by HW#5.1).

(a) Prove that for every $x \in X$, the subset $\{x\} \times Y = \{(x, y) : y \in Y\}$ of $X \times Y$ is isometric to Y. Likewise for every $y \in Y$, the subset $X \times \{y\} = \{(x, y) : x \in X\}$ is isometric to X.

(b*) Prove that if X and Y are both connected, then $X \times Y$ connected.

7. The goal of this problem is to prove that any open subset of $\mathbb{R}$ (with standard metric) is a **disjoint** union of at most countably many open intervals.

So, let U be any open subset of $\mathbb{R}$.

(a) Define the relation $\sim$ on U by setting $x \sim y \iff x = y$ or ($x < y$ and $[x, y] \subset U$) or ($y < x$ and $[y, x] \subset U$). Prove that $\sim$ is an equivalence relation.

(b*) Let A be an equivalence class with respect to $\sim$. Show that A is an open interval.

(c*) Deduce from (b) that U is a disjoint union of open intervals. Then prove that the number of those intervals is at most countable.

8. Use Problem 6 to show that the analogue of Problem 7 does not hold in $\mathbb{R}^2$, that is, there exist open subsets of $\mathbb{R}^2$ which are not representable as disjoint unions of open discs (an open disc is an open ball in $\mathbb{R}^2$).

Comment on 1: Let $\{a_{n,k}\}_{n,k\in\mathbb{N}}$ be a "double-index sequence" of real numbers, that is, a collection of real numbers indexed by two independent parameters n and k each of which ranges over $\mathbb{N}$. Consider the following two conditions on $\{a_{n,k}\}$:

(i) for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $|a_{n,k}| < \varepsilon$ for all $n, k \geq M$

(ii) $\lim\limits_{k\to\infty}(\lim\limits_{n\to\infty} a_{n,k}) = 0$.

Note that if $\{x_n\}$ is a sequence in some metric space (X, d) and we set $a_{n,k} = d(x_n, x_k)$ for all $n, k \in \mathbb{N}$, then (i) holds $\iff$ $\{x_n\}$ is Cauchy (by definition), and (ii) is the condition from Problem 1 in HW#5.

In general, (i) always implies (ii) (the proof is identical to that of Problem 1(a) in HW#5), but (ii) does not imply (i) in general. For instance, let $a_{n,k} = \frac{k}{n}$. Then for any k we clearly have $\lim\limits_{n\to\infty} a_{n,k} = 0$, whence $\lim\limits_{k\to\infty}(\lim\limits_{n\to\infty} a_{n,k}) = 0$, so (ii) holds. On the other hand, (i) does not hold since, for instance, $a_{n,n} = 1$ for all n.

Problem 1(b) asserts that (ii) does imply (i) in the special case when $a_{n,k} = d(x_n, x_k)$ for some sequence $\{x_n\}$ in a metric space. In view of the above example, it is clear that one must use some properties of the distance function (that is, axioms of a metric space) in order to prove this implication.

Hint for 2: The "$\Leftarrow$" direction is easy. For the "$\Rightarrow$" direction, assume that $X = A \sqcup B$ with A, B closed, and show that the function $f : X \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B \end{cases}$$

is continuous.

Hint for 3(a): Use Problem 2 in Homework#3.

Hint for 3(b): Use (a) and Theorem 11.1 from class (=Theorem 4.22 from Rudin).

Hint for 4: Assume that X is disconnected and use 3(a) to reach a contradiction.

Hint for 6(b): By 5 and 6(a) all subsets of the form $X \times \{y\}$ and $\{x\} \times Y$ are connected. Start with this fact and use Problem 4 twice. Drawing a picture in the case $X = Y = [0, 1]$ will likely be helpful.

Hint for 7(b): First prove that A has Intermediate Value Property (as defined in class), and therefore A is an interval by Theorem 11.2 from class. Then assume that A is not an open interval and reach a contradiction.

Hint for 7(c): Use the fact that $\mathbb{Q}$ is countable.