**Homework #5. Due Thursday, October 1st, in class**

**Reading:**

1. For this homework assignment: Sections 4.1-4.3 (limits of functions, continuous functions, continuity and compactness) + class notes (Lectures 9-10).
2. For next week's classes: 2.5 (connected sets), 4.4 (continuity and connectedness). We will also talk about contraction mappings and fixed points (not explicitly discussed in Rudin). Note that Rudin gives a rather non-standard definition of a connected metric space. You can look up the standard definition (which we will introduce in class) on wikipedia (article name: connected space).

**Problems:**

**Note on hints:** All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

**1.** Let $\{x_n\}$ be a sequence in a metric space $(X, d)$.

(a)* Assume that $\{x_n\}$ is Cauchy. Prove that $\lim\limits_{k\to\infty}(\lim\limits_{n\to\infty} d(x_k, x_n)) = 0$. In other words, prove that
  (i) for every $k \in \mathbb{N}$ the limit $\lim\limits_{n\to\infty} d(x_k, x_n)$ exists
  (ii) if we set $a_k = \lim\limits_{n\to\infty} d(x_k, x_n)$, then $\lim\limits_{k\to\infty} a_k = 0$.

(b) (bonus) Prove that the converse of (a) is also true: if $\lim\limits_{k\to\infty}(\lim\limits_{n\to\infty} d(x_k, x_n)) = 0$, then the sequence $\{x_n\}$ is Cauchy.

**2*.** Prove Theorem 9.1 from class: let $Z$ be a metric space and let $Y$ be a dense subset of $Z$. Suppose that every Cauchy sequence in $Y$ converges in $Z$. Then $Z$ is complete.

**3.** Let $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ be a finite collection of metric spaces. Let $X = \prod_{k=1}^n X_k$ (here the product denotes the Cartesian product of sets), and define the function $d : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d((x_1, x_2, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{k=1}^{n} d_k(x_k, y_k)$$

(a) Prove that $(X, d)$ is a metric space. Note that if each $X_k = \mathbb{R}$ with standard metric, then $X = \mathbb{R}^n$ with Manhattan metric

(b) For each $1 \leq k \leq n$ define the projection map $\pi_k : X \to X_k$ by $\pi((x_1, \ldots, x_n)) = x_k$. Prove that $\pi_k$ is continuous.

(c)* Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function, that is, $f$ is a sum of finitely many functions $f_1, f_2, \ldots, f_m$ where each $f_i$ has the form $f_i(x_1, \ldots, x_n) = \alpha x_1^{d_1} \ldots x_n^{d_n}$ for some $\alpha \in \mathbb{R}$ and $d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}$ (for instance, if $n = 2$, we can have $f(x_1, x_2) = 1 + 3x_1 + 4x_1x_2 + x_2^3$). Prove that $f$ is continuous.

**4.** Let $X$ be an arbitrary metric space and $f : X \to \mathbb{R}$ a continuous function (where $\mathbb{R}$ is equipped with standard metric).

(i) Prove that the sets $\{x \in X : f(x) > 0\}$ and $\{x \in X : f(x) < 0\}$ are open and the set $\{x \in X : f(x) = 0\}$ is closed

(ii) Prove that if $g : X \to \mathbb{R}$ is another continuous function, then the set $\{x \in X : f(x) = g(x)\}$ is closed

**5.**

(a) Let $X$ be any set with discrete metric ($d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$). Prove that for any metric space $Y$, any function $f : X \to Y$ is continuous.

(b) Use (a) to show that there exist metric spaces $X$ and $Y$ and a function $f : X \to Y$ such that $f$ is continuous and bijective, but $f^{-1} : Y \to X$ is not continuous (recall that we proved in class that this cannot happen if $X$ is compact).

**6.** This problem outlines a different proof of Theorem 10.5 from class.

**Theorem 10.5:** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and assume that $X$ is compact. Then any continuous function $f : X \to Y$ is uniformly continuous.*

**Outline:** Suppose that $f : X \to Y$ is continuous, but not uniformly continuous. First show that there exist $\varepsilon > 0$ and sequences $\{a_n\}$ and $\{b_n\}$ in $X$ such that $d_X(a_n, b_n) < \frac{1}{n}$, but $d_Y(f(a_n), f(b_n)) \geq \varepsilon$ for all $n$. Since $X$ is compact, it is sequentially compact, so there exists a subsequence $\{a_{n_k}\}$ which converges to some $a \in X$. Use the fact that $d_X(a_n, b_n) \to 0$ as $n \to \infty$ to deduce that the sequence $\{b_{n_k}\}$ converges to $a$ as well. Now use Theorem 9.2 (sequential characterization of continuity) to reach a contradiction with the assumption that $d_Y(f(a_n), f(b_n)) \geq \varepsilon$ for all $n$.

**Hint for 1(a):** Part (i) follows from one of the problems in HW#4. For part (ii) use the fact that if $\{b_n\}$ is a convergent sequence of real numbers and there exist $M \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $|b_n| \leq C$ for all $n \geq M$, then $|\lim_{n\to\infty} b_n| \leq C$.

**Hint for 2:** Show that for any sequence $\{z_n\}$ in $Z$ there is a sequence $\{y_n\}$ in $Y$ such that $d(y_n, z_n) \to 0$ as $n \to \infty$; then show that if $\{z_n\}$ is Cauchy, then $\{y_n\}$ is also Cauchy.

**Hint for 3(c):** Use 3(b) and a suitable theorem from Rudin.

**Hint for 5:** You can construct an example where $X = Y$ as sets (but with different metrics) and $f : X \to Y$ is the identity function ($f(x) = x$ for all $x$).