Homework #4. Due Thursday, September 24th, in class

Reading:

1. For this homework assignment: Sections 2.3 (compact sets), 3.1-3.3 (convergent sequences, subsequences, Cauchy sequences) + class notes (Lectures 7-8).
2. For next week's classes: 4.1-4.2 (limits of functions, continuity). Before we get to continuity we will finish the discussion of completions of metric spaces (started at the end of Lecture 8) and then talk about contraction mappings and fixed points (not explicitly discussed in Rudin).

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

1. (Lemma 7.6 from class) Let $\{x_n\}$ be a sequence in a metric space (X, d), and let x be some element of X. Prove that the following conditions are equivalent:

(i) some subsequence of $\{x_n\}$ converges to x
(ii) for every $\varepsilon > 0$ there are infinitely many n for which $x_n \in N_\varepsilon(x)$.

When proving the implication (ii)$\Rightarrow$(i) make it clear how you use that $x_n \in N_\varepsilon(x)$ for infinitely many n (and not just for some n).

2*. Let (X, d) be a metric space and Y a subset of X.

(a) Suppose that X is sequentially compact and Y is a closed subset of X. Prove that Y is also sequentially compact.
(b) Now assume that Y is sequentially compact. Prove that Y is closed in X (we are not assuming anything about X).

Note: Since sequential compactness is equivalent to compactness for metric spaces, the results of parts (a) and (b) of this problem are equivalent to Theorems 6.1 and 6.2 from class (=Theorems 2.35 and 2.34 in Rudin), respectively. The point of this exercise is to prove the results working directly with the definition of sequential compactness.

3. Let $X = C[a, b]$, considered as a metric space with uniform metric d_{unif} (as defined in Problem 4 in HW#2). Prove that the set $B_1(\mathbf{0})$, the closed ball of radius 1 centered at $\mathbf{0}$ in X, is not sequentially compact (and hence not compact).

4. Recall the concept of the **completion** of a metric space introduced in Lecture 8. In the statement of the problem we will use the notion of isometric metric spaces. Metric spaces (X, d) and (X', d') are called isometric (to each other) if there is a bijection $f : X \to X'$ such that $d(x, y) = d'(f(x), f(y))$ for all $x, y \in X$.

Let (X, d) be a metric space. Let $\Omega(X)$ be the set of all Cauchy sequences $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in X$ for each n. Define the relation $\sim$ on Ω by setting

$$\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

(a) Prove that $\sim$ is an equivalence relation.

Now let $\widehat{X} = \Omega/\sim$, the set of equivalence classes with respect to $\sim$. The equivalence class of a sequence $\{x_n\}$ will be denoted by $[x_n]$. For instance, $[\frac{1}{n}] = [\frac{1}{n^2}]$ since the sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n^2}$ are equivalent. Given an element $x \in X$, we will denote by $[x] \in \widehat{X}$ the equivalence class of the constant sequence all of whose elements are equal to x.

Now define the function $D : \widehat{X} \times \widehat{X} \to \mathbb{R}_{\geq 0}$ by setting

$$D([x_n], [y_n]) = \lim_{n\to\infty} d(x_n, y_n) \qquad (***)$$

(b)* Prove that the limit on the right-hand side of (***) always exists and that the function D is well-defined (that is, if $[x_n] = [x'_n]$ and $[y_n] = [y'_n]$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$).

(c) Prove that $(\widehat{X}, D)$ is a metric space

(d) Consider the map $\iota : X \to \widehat{X}$ given by $\iota(x) = [x]$ (that is, ι sends each x to the equivalence class of the corresponding constant sequence). Prove that ι is injective and $D(\iota(x), \iota(y)) = d(x, y)$ for all $x, y \in X$. This implies that (X, d) is isometric to the metric space $(\iota(X), D)$ (so identifying X with $\iota(X)$, we can think of X as a subset of $(\widehat{X}, D)$).

5. A metric space (X, d) is called **ultrametric** if for any $x, y, z \in X$ the following inequality holds:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(Note that this inequality is much stronger than the triangle inequality). If X is any set and d is the discrete metric on X (that is, $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$), then clearly (X, d) is ultrametric. A more interesting example of an ultrametric space is given in the next problem.

Prove that properties (i) and (ii) below hold in any ultrametric space (X, d) (note that both properties are counter-intuitive since they are very far from being true in $\mathbb{R}$).

(i) Take any $x \in X$, $\varepsilon > 0$ and take any $y \in N_\varepsilon(x)$. Then $N_\varepsilon(y) = N_\varepsilon(x)$. This means that if we take an open ball of fixed radius around some point x, then for any other point y from that open ball, the open ball of the same radius, but now centered at y, coincides with the original ball. In other words, any point of an open ball happens to be its center.

(ii) Prove that a sequence $\{x_n\}$ in X is Cauchy $\iff$ for any $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that $d(x_{n+1}, x_n) < \varepsilon$ for all $n \geq M$. **Note:** The forward implication holds in any metric space.

6. Let p be a fixed prime number. Define the function $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ as follows: given a nonzero $x \in \mathbb{Q}$, we can write $x = p^a \frac{c}{d}$ for some $a, c, d \in \mathbb{Z}$ where c and d are not divisible by p . Define $|x|_p = p^{-a}$ (note that the above representation is not unique, but it is easy to see that a is uniquely determined by x). For instance,

$$\left|\frac{9}{20}\right|_p = \begin{cases} \frac{1}{9} & \text{if } p = 3 \\ 4 & \text{if } p = 2 \\ 5 & \text{if } p = 5 \\ 1 & \text{for any other } p. \end{cases}$$

Also define $|0|_p = 0$. Now define the function $d_p : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by $d_p(x, y) = |y - x|_p$.

(a) Prove that $(\mathbb{Q}, d_p)$ is an ultrametric space. (Note: the completion of this metric space is usually denoted by $\mathbb{Q}_p$ is called *p-adic numbers*).

(b) Describe explicitly the set $N_1(0)$ (the open ball of radius 1 centered at 0) in $(\mathbb{Q}, d_p)$.

(c) Let d be the standard metric on $\mathbb{Q}$ (that is, $d(x, y) = |y - x|$ where $|\cdot|$ is the usual absolute value). Give examples of sequences $\{x_n\}$ and $\{y_n\}$ in $\mathbb{Q}$ such that

(i) $x_n \to 0$ in $(\mathbb{Q}, d_p)$ but $\{x_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d)$

(ii) $y_n \to 0$ in $(\mathbb{Q}, d)$ but $\{y_n\}$ is unbounded as a sequence in $(\mathbb{Q}, d_p)$

Hint for #2: Use sequential characterization of closures (Lemma 7.4 from class) and Theorem 5.1(a). For (b) also use that a subsequence of a convergent sequence converges (to the same limit as the entire sequence).

Hint for #4(b): For the existence of the limit prove that the sequence $\{d(x_n, y_n)\}$ is Cauchy using the inequality $d(x, w) \leq d(x, y) + d(y, z) + d(z, w)$.