**Homework #3. Due Thursday, September 17th, in class**

**Reading:**

1. For this homework assignment: Sections 2.2 (metric spaces) and 2.3 (compact sets).
2. For next week's classes: 2.3 (compact sets), 3.1-3.3 (convergent sequences, subsequences, Cauchy sequences). Note that most of the material in 3.1-3.3 is essentially a review of topics from 3310, so we will go over these sections very quickly. Next week we also talk about sequential compactness and completions of metric spaces which are not discussed in the main text in Rudin.

**Problems:**

**Note on hints:** All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

**1.** Given a metric space $(X, d)$, a point $x \in X$ and $\varepsilon > 0$, define $B_\varepsilon(x) = \{y \in X : d(y, x) \leq \varepsilon\}$, called the *closed ball of radius* $\varepsilon$ *centered at* $x$.

(a) Prove that $B_\varepsilon(x)$ is always a closed subset of $X$.
(b) Deduce from (a) that $\overline{N_\varepsilon(x)} \subseteq B_\varepsilon(x)$, that is, the closure of the open ball of radius $\varepsilon$ centered at $x$ is contained in the respective closed ball.
(c) Is it always true that $\overline{N_\varepsilon(x)} = B_\varepsilon(x)$? Prove or give a counterexample.

**2.** Let $X$ be metric space, and let $Z \subseteq Y$ be subsets of $X$. Prove that $Z$ is closed as a subset of $Y \iff Z = Y \cap K$ for some closed subset $K$ of $X$. Deduce that if $Z$ is closed in $X$, then $Z$ is closed in $Y$. **Note:** The analogous result with closed sets replaced by open sets was proved in class (Lemma 5.2) and also appears as Theorem 2.30 in Rudin.

**3.** Let $(X, d)$ be a metric space and $S$ a subset of $X$. Prove that the following three conditions are equivalent (as defined in class, $S$ is called *bounded* if it satisfies either of those conditions):

(i) There exists $x \in X$ and $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(ii) For any $x \in X$ there exists $R \in \mathbb{R}$ such that $S \subseteq N_R(x)$.
(iii) The set $\{d(s, t) : s, t \in S\}$ is bounded above as a subset of $\mathbb{R}$.

**4.** Let $K = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Prove that $K$ is compact in two different ways:

(i) by showing that $K$ is closed and bounded as a subset of $\mathbb{R}$ (We will prove in class next week that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded).

(ii)* directly from definition of compactness.

**5*.** Let $X$ be a metric space. Prove that $X$ is compact $\iff$ $X$ satisfies the following property:

> Let $\{K_\alpha\}$ be any collection of closed subsets of $X$ such that for any finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_n}$, the intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_n}$ is non-empty. Then the intersection of all sets in $\{K_\alpha\}$ is non-empty.

**6.** Let $X$ be a set, and let $d_1$ and $d_2$ be two different metrics on $X$. Given $x \in X$ and $\varepsilon > 0$, define $N^1_\varepsilon(x) = \{y \in X : d_1(y, x) < \varepsilon\}$, the $\varepsilon$-neighborhood of $x$ with respect to $d_1$, and similarly define $N^2_\varepsilon(x) = \{y \in X : d_2(y, x) < \varepsilon\}$. We will say that $d_1$ and $d_2$ are *topologically equivalent* if a subset $S$ of $X$ is open with respect to $d_1$ $\iff$ it is open with respect to $d_2$. (**Note:** for brevity, if $d$ is a metric on $X$, we will say that $S$ is $d$-open if $S$ is open as a subset of the metric space $(X, d)$).

(a) Prove that $d_1$ and $d_2$ are topologically equivalent if and only if for every $\varepsilon > 0$ and every $x \in X$ there exist $\delta_1, \delta_2 > 0$ (depending on both $\varepsilon$ and $x$) such that $N^1_{\delta_1}(x) \subseteq N^2_\varepsilon(x)$ and $N^2_{\delta_2}(x) \subseteq N^1_\varepsilon(x)$.

(b) Suppose that there exist real numbers $A, B > 0$ such that $d_1(x, y) \leq Ad_2(x, y)$ and $d_2(x, y) \leq Bd_1(x, y)$ for all $x, y \in X$. Use (a) to prove that $d_1$ and $d_2$ are topologically equivalent.

(c) Now use (b) to prove that the Euclidean and Manhattan metrics on $\mathbb{R}^n$ are topologically equivalent.

**Definition:** Let $(X, d)$ be a metric space and $\varepsilon > 0$. A subset $S$ of $X$ is called an *$\varepsilon$-net* if for any $x \in X$ there exists $s \in S$ such that $d(x, s) < \varepsilon$. In other words, $S$ is an $\varepsilon$-net if $X$ is the union of open balls of radius $\varepsilon$ centered at elements of $S$.

**7*.** Let $S$ be a subset of a metric space $(X, d)$. Prove that the following are equivalent:

(i) The closure of $S$ is the entire $X$;

(ii) $U \cap S \neq \emptyset$ for any non-empty open subset $U$ of $X$;

(iii) $S$ is an $\varepsilon$-net for every $\varepsilon > 0$.

The subset $S$ is called *dense* (in $X$) if it satisfies these equivalent conditions.

**Hint for #4(ii):** Let $\{U_\alpha\}$ be any open cover of $K$. One of the $U_\alpha$ must contain 0. What can you say about that $U_\alpha$?

**Hint for #5:** There is a natural bijection between open covers of $X$ and collections of closed subsets with empty intersection.

**Hint for #7:** Prove that negations of (i), (ii) and (iii) are equivalent to each other