Homework #2. Due Thursday, September 10th, in class

Reading:

1. For this homework assignment: Sections 2.1 (finite, countable and uncountable sets) and 2.2 (metric spaces).
2. For next week's classes: 2.2 (metric spaces) and 2.3 (compact sets)

Problems:

Note on hints: All hints are given at the end of the assignment, each on a separate page. Problems (or parts of problems) for which hint is available are marked with *.

Definition: Two sets X and Y are said to have the same cardinality if there is a bijection from X to Y.

1. Let A be an uncountable set and B a countable subset of A.

(a) Prove that $A \setminus B$ is uncountable.

(b)* Prove that A and $A \setminus B$ have the same cardinality.

2*. Let X and Y be any sets, and define X^Y to be the set of all functions $f : Y \to X$. Prove that if $|X| \geq 2$, then Y and X^Y do not have the same cardinality.

3. Rudin 2.2.

4. Let $a \leq b$ be real numbers and $X = C[a, b]$, the set of all continuous functions from $[a, b]$ to $\mathbb{R}$. Define the functions $d_{unif} : X \times X \to \mathbb{R}_{\geq 0}$ and $d_{int} : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d_{unif}(f, g) = \max_{x \in X} |f(x) - g(x)| \text{ and } d_{int}(f, g) = \int_a^b |f(x) - g(x)|\, dx.$$

(a) Prove that (X, d_{unif}) is a metric space (the metric d_{unif} is called the **uniform metric**)

(b) (practice) Prove that (X, d_{int}) is a metric space (the metric d_{int} is called the **integral metric**)

5. Let (X, d) be a metric space and S is a subset of X. Prove that S is open $\iff$ S is the union of some collection of open balls (which could be centered at different points).

6. Let (X, d) be a metric space and S a subset of X.

(i) Recall from Lecture 4 that a point $x \in X$ is called a *contact point of* S if $N_\varepsilon(x) \cap S \neq \emptyset$ for all $\varepsilon > 0$

(ii) A point $x \in X$ is called an *interior point of* S if there exists $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$

The set of all contact points of S is denoted by $\overline{S}$ and the set of all interior points of S is denoted by S^o.

(a*) Prove that the set S^o is always open.

(b) Let $x \in S$. Prove that x is a contact point of S $\iff$ x is not an interior point of $X \setminus S$.

(c) Use (a) and (b) to prove that the set $\overline{S}$ is always closed (using the definition of a closed set given in class).

7*. Let $X = C[a, b]$ and $d = d_{unif}$ (as defined in Problem 4). Find an (infinite) sequence $f_1, f_2, \ldots$ of elements of X such that $d(f_i, f_j) = 1$ for all $i \neq j$.

Hint for 1(b): Choose any countable subset C of $A \setminus B$ and then use things proved in class to show that the identity map $f : (A \setminus B) \setminus C \to (A \setminus B) \setminus C$ can be extended to a bijection from $A \setminus B$ and A. Draw a picture!

Hint for 2: Note that if $X = \{0, 1\}$ and $Y = \mathbb{N}$, then X^Y is precisely the set of infinite sequences of 0 and 1, so the assertion of the problem in this special case holds by Theorem 3.5. To prove the general case imitate the proof of Theorem 3.5.

Hint for 6(a): Use the fact that open balls are open sets.

Hint for 7: As observed in class, if we replace $C[a, b]$ by $B[a, b]$, the set of all bounded functions from $[a, b]$ to $\mathbb{R}$ and define the metric d on $B[a, b]$ by $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$, then the analogous question would have a very simple answer, e.g. we could let $f_n = I_{1/n}$ where I_c (for a fixed $c \in \mathbb{R}$) is the function defined by $I_c(c) = 1$ and $I_c(x) = 0$ for $x \neq c$. To solve Problem 7 think of a suitable way to approximate I_c by a continuous function.