Homework #12. Not due

Reading:

For this homework assignment: Sections 28 and 29.1 from Kolmogorov-Fomin (class handout) + class notes (Lectures 25-27); alternative reference is Rudin, Sections 11.4 (measurable functions), 11.5 (simple functions) and beginning of 11.6 (integration).

Let X be a metric space and A a subset of X. Recall that

- A is called *nowhere dense* if for every non-empty open subset U of X there exists a non-empty open subset V of U with $V \cap A = \emptyset$.
- A is a called an F_σ-set if A is countable union of closed sets.
- A is a called a G_δ-set if A a countable intersection of open sets.

Problems:

1*. Let X be a metric space and A a subset of X. Prove that the following four conditions are equivalent:

(i) A is nowhere dense

(ii) for every non-empty open ball U in X there exists a non-empty open ball V contained in U with $V \cap A = \emptyset$.

(iii) $X \setminus A$ contains an open dense subset of X

(iv) $(\overline{A})^o = \emptyset$, that is, the closure of A has empty interior (interior is a shortcut for "the set of interior points"). Recall that the notion of an interior point of a subset was defined in Homework #2.

2. Let X be a metric space.

(i) Let A be a subset of X. Prove that A is an F_σ-set $\iff$ $X \setminus A$ is a G_δ-set.

(ii) Prove that the collection of all F_σ-sets in X is closed under countable unions and the collection of all G_δ-sets in X is closed under countable intersections.

(iii)* Let $X = \mathbb{R}$ (with standard metric). Prove that every open subset of X is an F_σ-set and every closed subset of X is a G_δ-set.

3. Let $D : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function (defined by $D(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$), and let $a < b$ be real numbers.

(a) Prove that D is Lebesgue-integrable on $[a, b]$ and that $\int_{[a,b]} D \, d\mu = 0$.

(b) Prove that D is not Riemann-integrable on $[a, b]$.

4*. Rudin, Problem 3 after Chapter 11 (p. 332).

5. Kolmogorov-Fomin, Problem 6 after Section 28 (p.292)

6. Kolmogorov-Fomin, Problem 8 after Section 28 (p.292). **Note:** The functions $f_i^{(k)}$ are only defined for $1 \leq i \leq k$. It is probably useful to start by drawing the graphs of the first few functions in the sequence (say for $k = 1, 2, 3$).

Hint for 1: It is probably the most convenient to first prove (i) $\iff$ (ii) and then (i)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i).

Hint for 2(iii): Start by showing that every open interval is an F_σ-set. Once this is done, the rest follows by direct combination of previously known results.

Hint for 4: Use Cauchy criterion to express the set in question in terms of sets of the form $\{x : |f_n(x) - f_m(x)| < \frac{1}{k}\}$ using countable unions and countable intersections.