Homework #11. Due Thursday, December 3rd, in class

Reading:

1. For this homework assignment: Section 25.1 from Kolmogorov-Fomin (class handout) + class notes (Lectures 22-25); alternative reference is Rudin, Section 11.2 (construction of the Lebesgue measure).

2. For the last 4 classes: TBA

Problems:

1. Let A_1, A_2, B_1 and B_2 be subsets of the same set. Prove that

(a) $(A_1 \cup A_2) \triangle (B_1 \cup B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

(b) $(A_1 \cap A_2) \triangle (B_1 \cap B_2) \subseteq (A_1 \triangle B_1) \cup (A_2 \triangle B_2)$

Recall that both formulas were used in verification of properties of Lebesgue measure.

2*. Problem 1 from Kolmogorov-Fomin (p. 268). Note that this is a problem about subsets of $\mathbb{R}^2$ (not $\mathbb{R}$).

3.

(a) Let A be a countable subset of $[0,1]$. Prove that A has measure zero (that is, A is measurable and $\mu(A) = 0$).

(b) Prove that the Cantor set C has measure 0.

4*. Let A be a subset of $[0,1]$, and suppose that for every $\varepsilon > 0$ there exists a measurable set B such that $\mu^*(A \triangle B) < \varepsilon$. Prove that A is measurable.

5. The main goal of this problem is to justify the definition of the Cantor staircase function given in class.

Let $a \leq b$ be real numbers, let S be a subset of $[a,b]$ which contains both a and b, and let $f : S \to \mathbb{R}$ be an increasing function (note: f is only defined on S).

(a)* Prove that f can be extended to an increasing function from $[a,b]$ to $\mathbb{R}$, that is, there exists an increasing function $F : [a,b] \to \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in S$.

(b)* Now assume that $f(S)$ is a dense subset of $[f(a), f(b)]$. Prove that there exists unique function F satisfying the conclusion of (a). Then prove that this F is continuous.

6 (bonus) Let $F : [0,1] \to [0,1]$ be the Cantor staircase function and C the Cantor set. The argument from Lecture 24 shows that $\mu(F(C)) = 1$ (while

$\mu(C) = 0$). Modify the construction of F to show that for every $\varepsilon > 0$ there exists a **strictly** increasing continuous function $F_\varepsilon : [0,1] \to [0,1]$ such that $\mu(F_\varepsilon(C)) > 1 - \varepsilon$.

7. Problem 7 from Kolmogorov-Fomin (p. 268). Note that the hint given in KF is essentially a sketch of the solutions. The things you need to justify are

(a) $C = \cup_{n=-\infty}^{\infty} \Phi_n$
(b) $\Phi_n \cap \Phi_m = \emptyset$ if $n \neq m$
(c) Assume that Φ_0 is measurable. Then each Φ_n is measurable and $\mu(\Phi_n) = \mu(\Phi_0)$ for all $n \in \mathbb{Z}$
(d) the conclusion of (c) contradicts (33) in KF.

Remark: The Lebesgue measure on the circle C can be defined in exactly the same way as on $[0,1]$ with the exception that we call a subset of C elementary if it is a finite union of arcs.

Hint for 2(a): Show that any open subset of $\mathbb{R}^2$ can be written as a union of squares whose endpoints have rational coordinates.

Hint for 4: Think of $\mu^*(A \triangle B)$ as the distance between A and B and use a suitable formula on page 306 from Rudin. Note that Rudin's notation for $A \triangle B$ is $S(A, B)$.

Hint for 5(a): Define F by an explicit formula involving the supremum of values of f on a suitable subset.

Hint for 5(b): Prove uniqueness by contradiction. Continuity immediately follows from an earlier homework problem.