Homework #10. Due Thursday, November 12th, in class

Reading:

1. For this homework assignment: Kolmogorov-Fomin 8.2 (Picard's theorem) and Rudin 7.7 (the Stone-Weierstrass Theorem) + class notes (Lectures 19-20)

2. The proof of Stone-Weierstrass theorem (Theorem 7.32 in Rudin) + review of Riemann integral (beginning of Section 6.1 in Rudin).

Problems:

1. Let X be a compact metric space.

(a) Let $a \in X$. Prove that the evaluation map $eval_a : C(X) \to \mathbb{R}$ given by $eval_a(f) = f(a)$ is continuous

(b)* Let K be a closed subset of $\mathbb{R}$ and let $\Omega = \{f \in C(X) : f(X) \subseteq K\}$. Use (a) and characterization of continuity in terms of closed sets to prove that Ω is a closed subset of $C(X)$.

Note that a special case of (b) was used in the proof of Picard's Theorem in Lecture 19.

2. Let $G = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 25\}$, and define $f : G \to \mathbb{R}$ by

$$f(x,y) = \sqrt{25 - x^2 - y^2} + \sin(x).$$

Use the proof of Picard's Theorem from Kolmogorov-Fomin to show that the differential equation $\frac{dy}{dx} = f(x,y)$ has a solution $y = \phi(x)$ defined on the interval $[\frac{7}{4}, \frac{9}{4}]$ satisfying the initial condition $\phi(2) = 2$. Partial credit will be given for proving the same thing for the interval $[2-\delta, 2+\delta]$ for some (explicit) $\delta < \frac{1}{4}$. **Note:** The proof given in class is essentially the same as the one in KF, but specific bounds on δ from the proof in class are not sufficient to get $\delta = \frac{1}{4}$.

3. Let $a < b$ be real numbers and let $\mathcal{P}_{even}[a,b] \subseteq C[a,b]$ be the set of all even polynomials (that is, polynomials which only involve even powers of x).

(a) Use Stone-Weierstrass Theorem to prove that $\mathcal{P}_{even}[a,b]$ is dense in $C[a,b] \iff 0 \notin (a,b)$.

(b)* (optional) Now prove the "$\Leftarrow$" direction in (a) using only Weierstrass Approximation Theorem (but not Stone-Weierstrass Theorem).

4.

(a)* Prove that the (direct) analogue of Weierstrass Approximation Theorem does not hold for $C(\mathbb{R})$, continuous functions from $\mathbb{R}$ to $\mathbb{R}$: Show that there exists $f \in C(\mathbb{R})$ which cannot be uniformly approximated by polynomials, that is, there is no sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $\mathbb{R}$.

(b)* Now prove that the following (weak) version of Weierstrass Approximation Theorem holds for $C(\mathbb{R})$: for any $f \in C(\mathbb{R})$ there exists a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[a, b]$ for any closed interval $[a, b]$ (of course, the point is that a single sequence will work for all intervals).

Hint for 1: Represent Ω as the intersection of certain (possibly infinite) collection of sets and use (a) and characterization of continuity in terms of closed sets to show that each set in that collection is closed.

Hint for 3(b): WOLOG assume that $0 \leq a < b$. Start by showing that any continuous function in $g \in C[a, b]$ can be written as $g(x) = h(x^2)$ for some continuous function $h \in C[a^2, b^2]$.

Hint for 4(a): Use the fact that any non-constant polynomial $p(x)$ tends to $\pm\infty$ as $x \to \infty$.

Hint for 4(b): It is enough to prove the result for intervals of the form $[-k, k]$ for $k \in \mathbb{N}$ (why?). To construct a sequence of polynomials $\{p_n\}$ s.t. $p_n \rightrightarrows f$ on $[-k, k]$ for each k, apply Weierstrass Approximation Theorem on each interval and then use a diagonal-type argument.