Homework #1. Due Thursday, September 3rd, in class

Reading:

1. For this homework assignment: Chapter 1.

2. Before the class on Tue, Sep 1: Section 2.1 (finite, countable and uncountable sets). Before the class on Thu, Sep 3: Section 2.2 (metric spaces).

Below Rudin x.y refers to Problem y after Chapter x in Rudin.

Problems:

1. Prove that $\mathbb{C}$ (complex numbers) cannot be made an ordered field (no matter how the order relation $<$ is defined). **Note:** in class we proved this for lexicographic order; the general proof is not much more complicated.

2. Rudin 1.5.

3. Let S be an ordered set and A and B subsets of S such that

(i) $a \le b$ for any $a \in A$ and $b \in B$;

(ii) $\sup(A)$ and $\inf(B)$ exist in S.

Prove that $\sup(A) \le \inf(B)$.

4. Let $(X, <)$ be an ordered set. A subset S of X will be called *basic* if the following is true: if $s \in S$ and $x \in X$ and $x < s$, then $x \in S$ (that is, for each element $s \in S$, all elements of X smaller than s also lie in S). For instance, any interval of the form $(-\infty, \alpha)$ or $(-\infty, \alpha]$ is a basic subset of $\mathbb{R}$.

(a) Given $a \in X$, define $D_<(a) = \{x \in X : x < a\}$ and $D_\le(a) = \{x \in X : x \le a\}$. Prove that the sets $D_<(a)$ and $D_\le(a)$ are basic.

(b) Let $S \subseteq X$ be a basic set, and suppose that $\sup(S)$ exists (in X; we do not assume that $\sup(S) \in S$). Prove that $S = D_<(\sup(S))$ or $S = D_\le(\sup(S))$. **Hint:** It is clear that $S \subseteq D_\le(\sup(S))$ for any set S such that $\sup(S)$ exists; suppose now that S is also basic and that S is different from $D_<(\sup(S))$ and $D_\le(\sup(S))$. This implies that there exists $y \in X$ such that $y < \sup(S)$ and $y \notin S$ (explain why). Now use the fact that S is basic to get a contradiction with the definition of $\sup(S)$.

In parts (c) and (d) of this problem $X = \mathbb{C}$ and $<$ is the lexicographic order relation on $\mathbb{C}$ defined as follows: $a + bi < c + di$ (with $a, b, c, d \in \mathbb{R}$) $\iff$ $a < c$ or ($a = c$ and $b < d$)

(c) For each $z = u + vi \in \mathbb{C}$ describe explicitly the set $D_<(z)$

(d) Find an (explicit) example of a non-empty subset S of $\mathbb{C}$ which is bounded above but such that $\sup(S)$ does not exist (and prove that your subset has the required property). **Hint:** According to part (b) it is enough to find a set S which is basic, bounded above and such that S is not equal to $D_<(z)$ or $D_\leq(z)$ for any $z \in \mathbb{C}$ (the latter condition can be verified using the answer in (c)).

5. Give a detailed and rigorous proof of the fact that

$$\lim_{n\to\infty} \frac{2n+3}{3n+4} = \frac{2}{3}$$

directly from the definition of limit of a sequence.

6. (practice) Let k be a positive integer, and let $x, y \in \mathbb{R}^k$, with $x \neq 0$ and $y \neq 0$. According to Theorem 1.37(e), we always have inequality $|x + y| \leq |x| + |y|$. Prove that equality holds $\iff$ $y = \lambda x$ for some scalar $\lambda > 0$. **Hint:** The backwards direction is easy. For the forward direction assume that $|x + y| = |x| + |y|$. First use the proof of Theorem 1.37 to deduce that $x \cdot y = |x||y|$. Then use the proof of the Schwartz inequality (Theorem 1.35 in Rudin) to show that $(x \cdot y)^2 = |x|^2|y|^2$ forces the vectors x and y to be proportional (that is, $y = \lambda x$ for some $\lambda \in \mathbb{R} \setminus \{0\}$).

7. **(bonus)** Deduce the Intermediate Value Theorem and Extreme Value Theorems directly from the following four results (which will be proved later in the course):

(1) Let $I = [a, b]$ be a closed bounded interval in $\mathbb{R}$, and consider I as a metric space with the standard metric ($d(x, y) = |x - y|$). Then I is compact and connected.

(2) Let $S \subseteq \mathbb{R}$ be a subset which is both compact and connected (again with respect to the standard metric). Then $S = \emptyset$ or $S = [a, b]$ for some $a, b \in \mathbb{R}$ with $a \leq b$.

(3) Let X and Y be metric spaces and $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected (as usual $f(X) = \{f(x) : x \in X\}$ is the image (=range) of f).

(4) Let X and Y be metric spaces and $f : X \to Y$ be a continuous function. If X is compact, then $f(X)$ is compact.

Note: The definitions of compactness and connectedness for metric spaces are given in Chapter 2 of Rudin, but they are not needed for this problem (all you need to know is that these are certain properties of metric spaces).

Hint for Problem 3: First show that $\sup A \leq b$ for every $b \in B$ – this follows directly from the definition of supremum.