20. Normal subgroups

20.1. **Definition and basic examples.** Recall from last time that if G is a group, H a subgroup of G and $g \in G$ some fixed element the set $gH = \{gh : h \in H\}$ is called a left coset of H.

Similarly, the set $Hg = \{hg : h \in H\}$ is called a right coset of H.

Definition. A subgroup H of a group G is called normal if $gH = Hg$ for all $g \in G$.

The main motivation for this definition comes from quotient groups which will be introduced in Lecture 22.

Let us now see some examples of normal and non-normal subgroups.

Example 1. *Let G be an abelian group. Then any subgroup of G is normal.*

Indeed, if G is abelian, then $gh = hg$ for all $g \in G$ and $h \in H$, so in particular, $gH = \{gh : h \in H\} = \{hg : h \in H\} = Hg$.

Example 2. *Let G be any group. Recall that the center of G is the set*

$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}.$$

By Homework#6.10, $Z(G)$ is a subgroup of G. For the same reason as in Example 1, $Z(G)$ is always a normal subgroup of G; moreover, any subgroup of $Z(G)$ is normal in G.

Example 3. *Any group G is a normal subgroup of itself. Indeed, if G is any group and $H = G$, then there is only one left and right coset of H in G (namely G itself), so $gG = Gg = G$ for all $g \in G$.*

Example 4. $G = S_3$, $H = \langle (1,2,3) \rangle = \{e, (1,2,3), (1,3,2)\}$.

Let $g = (1,2)$. Then

$$gH = \{(1,2), (1,2)(1,2,3), (1,2)(1,3,2)\} = \{(1,2), (2,3), (1,3)\}$$
$$Hg = \{(1,2), (1,2,3)(1,2), (1,3,2)(1,2)\} = \{(1,2), (1,3), (2,3)\}.$$

Note that while there exists $h \in H$ s.t. $gh \neq hg$, we still have $gH = Hg$ as sets.

The above computation does not yet prove that H is normal in G since we only verified $gH = Hg$ for a single g. To prove normality we would need to do the same for all $g \in G$. However, there is an elegant way to prove normality in this example, given by the following proposition.

Proposition 20.1. *Let G be a group and H a subgroup of index 2 in G. Then H is normal in G.*

Proof. This will be one of the problems in Homework#10. □

Recall from Lecture 19 that the index of H in G, denoted by $[G : H]$, is the number of left cosets of H in G and that if G is finite, then $[G : H] = \frac{|G|}{|H|}$. In Example 3 we have $|G| = 6$ and $|H| = 3$, so $[G : H] = 2$ and Proposition 20.1 can be applied.

Finally, we give an example of a non-normal subgroup:

Example 5. $G = S_3,\ H = \langle(1,2)\rangle = \{e, (1,2)\}$.

To prove this subgroup is not normal it suffices to find a single $g \in G$ such that $gH \neq Hg$. We will show that $g = (1,3)$ has this property.

We have $gH = \{(1,3), (1,3)(1,2)\} = \{(1,3), (1,2,3)\}$ and $Hg = \{(1,3), (1,2)(1,3)\} = \{(1,3), (1,3,2)\}$. Since $\{(1,3), (1,2,3)\} \neq \{(1,3), (1,3,2)\}$ (as sets), H is not normal.

20.2. **Conjugation criterion of normality.**

Definition. Let G be a group and fix $g, x \in G$. The element gxg^{-1} is called the conjugate of x by g.

Theorem 20.2 (Conjugation criterion)**.** *Let G be a group and H a subgroup of G. Then H is normal in $G \iff$ for all $h \in H$ and $g \in G$ we have $ghg^{-1} \in H$. In other words, H is normal in $G \iff$ for every element of H, all conjugates of that element also lie in H.*

Proof. "⇒" Suppose that H is normal in G, so for every element $g \in G$ we have $gH = Hg$. Hence for every $h \in H$ we have $gh \in gH = Hg$, so $gh = h'g$ for some $h' \in H$. Multiplying both sides on the right by g^{-1}, we get $ghg^{-1} \in H$. Thus, we showed that $ghg^{-1} \in H$ for all $g \in G, h \in H$, as desired.

"⇐" Suppose now for all $g \in G, h \in H$ we have $ghg^{-1} \in H$. This means that $ghg^{-1} = h'$ for some $h' \in H$ (depending on g and h). The equality $ghg^{-1} = h'$ can be rewritten as $gh = h'g$. Since $h'g \in Hg$ by definition, we get that $gh \in Hg$ for all $h \in H, g \in G$, so $gH \subseteq Hg$ for all $g \in G$.

Since the last inclusion holds for all $g \in G$, it will remain true if we replace g by g^{-1}. Thus, $g^{-1}H \subseteq Hg^{-1}$ for all $g \in G$. Using Lemma 19.1 (associativity of multiplication of subsets in a group), multiplying the last inclusion by g on both left and right, we get $Hg \subseteq gH$.

Thus, for all $g \in G$ we have $gH \subseteq Hg$ and $Hg \subseteq gH$, and therefore $gH = Hg$. □

20.3. **Applications of the conjugation criterion.**

Theorem 20.3. *Let G and G' be groups and $\varphi : G \to G'$ a homomorphism. Then* $\text{Ker}\,(\varphi)$ *is a normal subgroup of G.*

Proof. Let $H = \text{Ker}\,(\varphi)$. We already know from Lecture 16 that H is a subgroup of G, so it suffices to check normality. We will do this using the conjugation criterion.

So, take any $h \in H$ and $g \in G$. By definition of the kernel we have $\varphi(h) = e'$ (the identity element of G'). Hence $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)e'\varphi(g)^{-1} = e'$, so $ghg^{-1} \in \text{Ker}\,(\varphi) = H$. Therefore, H is normal by Theorem 20.2. □

Here are two more examples of application of the conjugation criterion

Example 6. *Let A and B be any groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\} \subseteq G$, the set of elements of G whose second component is the identity element of B.*

It is not hard to show that $\widetilde{A}$ is a subgroup of G and $\widetilde{A} \cong A$ (one can think of $\widetilde{A}$ as a canonical copy of A in G).

We claim that $\widetilde{A}$ is normal in G. Indeed, take any $g \in G$ and $h \in A$. Thus, $g = (x, y)$ and $h = (a, e_B)$ for some $a, x \in A$ and $y \in B$. Then $g^{-1} = (x^{-1}, y^{-1})$, so $ghg^{-1} = (x, y)(a, e_B)(x^{-1}, y^{-1}) = (xax^{-1}, ye_By^{-1}) = (xax^{-1}, e_B) \in \widetilde{A}$. Thus, $\widetilde{A}$ is normal by Theorem 20.2.

Example 7. *Let F be a field. Let*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F, ac \neq 0 \right\} \qquad \text{and} \qquad H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$$

In Lecture 12 we proved that G is a subgroup of $GL_2(F)$ (so G itself is a group). We also know that H is a subgroup $GL_2(F)$ (by Homework #7.7(a)); since clearly $H \subseteq G$, it follows that H is a subgroup of G.

Let us now use conjugation criterion to prove that H is normal in G. Take any $h \in H$ and $g \in G$. Then $g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some $a, b, c \in F$, with $a \neq 0, c \neq 0$ and $h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for some $x \in F$. Then

$$g^{-1}hg = (ac)^{-1} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = (ac)^{-1} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b + cx \\ 0 & c \end{pmatrix}$$
$$= (ac)^{-1} \begin{pmatrix} ac & c^2x \\ 0 & ac \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}cx \\ 0 & 1 \end{pmatrix} \in H,$$

so by the conjugation criterion H is normal in G.

Exercise 1. *Prove that H is NOT normal in $GL_2(F)$.*

To prove this it suffices to find two elements $h \in H$ and $g \in GL_2(F)$ such that $ghg^{-1} \notin H$.

20.4. **Book references.** The general references for this lecture are Section 4.5 in Gilbert and the second half of Chapter 14 in Pinter. Note that in Pinter, the conjugation criterion is used as the definition of a normal subgroup. The equivalence with the other definition (in terms of cosets) is established in Chapter 15, in the course of the discussion of quotient groups.