Homework #7. Due Thursday, March 24th

Reading:

1. For this assignment: Online lectures 13-15, [Pinter, §9-11] and [Gilbert, §3.4, 3.5].
2. For next week's classes: Online lectures 15 and 16, [Pinter, §9, 14] and [Gilbert, §3.5, 3.6]. Also read the definition of direct product (see a note 'Direct sums and products' on last year's webpage as well as Section G in the exercise after § 4 in Pinter).

Problems:

Problem 1: Recall that for a ring R with 1 we denote by $R^{\times}$ the group of invertible elements of R with respect to multiplication. For each of the following groups G, determine whether it is cyclic or not. If it is cyclic, find ALL generators (note: to prove that a group is cyclic it suffices to find one generator).

(i) $G = \mathbb{Z}_7^{\times}$, (ii) $G = \mathbb{Z}_9^{\times}$, (iii) $G = \mathbb{Z}_{12}^{\times}$.

Problem 2: Let x be an element of a group G, let $n = o(x)$, and assume that $n < \infty$. Prove that

$$o(x^k) = \frac{n}{gcd(n,k)}$$

for every $k \in \mathbb{Z}$ (this is a slight reformulation of Theorem 14.1(v) from online Lecture 14). You are allowed to use Theorem 14.1(iv) proved in online notes (but obviously not Theorem 14.1(v)).

Problem 3: (practice) Theorem 14.1 is applicable to any finite cyclic group G and any generator x of G. If $G = (\mathbb{Z}_n, +)$ for some n, we can use $x = [1]$ as a generator, in which case all assertions of the Theorem can be restated directly in terms of n. For instance, part (i) would say: "Every subgroup of $\mathbb{Z}_n$ is cyclic and is equal to $\langle [d] \rangle$ where d is a positive divisor of n". Restate other parts of Theorem 14.1 in a similar way.

Problem 4: Use restatement of Theorem 14.1 from Problem 3 to do the following:

(a) List all generators of $(\mathbb{Z}_{12}, +)$ and $(\mathbb{Z}_{15}, +)$
(b) List all subgroups of $(\mathbb{Z}_{12}, +)$ and $(\mathbb{Z}_{15}, +)$ (without repetitions)

Problem 5: (practice) Prove that the relation $\cong$ of "being isomorphic" is an equivalence relation (Claim 15.1 from online Lecture 15).

Hint: To prove that $\cong$ is symmetric, show that if $\varphi : G \to G'$ is an isomorphism, then the inverse map $\varphi^{-1} : G' \to G$ is also an isomorphism. Since the inverse of a bijection is a bijection, you only need to show that $\varphi^{-1}(uv) = \varphi^{-1}(u)\varphi^{-1}(v)$ for all $u, v \in G'$. To prove this, take any $u, v \in G'$, and let $x = \varphi^{-1}(u), y = \varphi^{-1}(v)$. Then $\varphi(x) = u$ and $\varphi(y) = v$; at this point you can use the fact that φ is a isomorphism.

Problem 6:

(a) Let $G = (\mathbb{Z}_6, +)$ and $G' = (\mathbb{Z}_7^{\times}, \cdot)$. Prove that $G' \cong G$ and find an explicit isomorphism $\varphi : G \to G'$. **Hint:** Use Theorem 15.2 from online notes.

(b) (practice) Use map φ from (a) to show (explicitly) that the multiplication tables of G and G' can be obtained from each other by relabeling of elements.

Problem 7: (practice)

(a) Let $G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$. Prove that G is a subgroup of $GL_2(\mathbb{R})$.

(b) Let G be the group from part (a). Find an isomorphism φ from $(\mathbb{R}, +)$ to G (and prove that φ is an isomorphism).

Problem 8: Let G be a group from Problem 2 in Homework #6: $G = \mathbb{R} \setminus \{-1\}$ as a set, and the operation $*$ on G is defined by $x * y = xy + x + y$. Prove that $(G, *)$ is isomorphic to $(\mathbb{R}\setminus\{0\}, \cdot)$ and find an explicit isomorphism between those groups.

Problem 9: Let $\varphi : G \to G'$ be an isomorphism, and let $g \in G$.

(a) Prove by induction that $\varphi(g^n) = \varphi(g)^n$ for every $n \in \mathbb{N}$.

(b) Prove that if $n \in \mathbb{N}$, then $g^n = e_G$ if and only if $\varphi(g)^n = e_{G'}$ (where e_G is the identity element of G and $e_{G'}$ is the identity element of G'). **Hint:** Use (a) and the fact that an isomorphism must send identity element to identity element (this will be proved in class next week).

(c) Use (b) to prove that $o(g) = o(\varphi(g))$ (Proposition 15.3 from online notes). Thus isomorphisms preserve orders of elements.

Problem 10: Let G be a group and $g, h \in G$.

(a) Prove that the elements ghg^{-1} and h have the same order by direct computation.

(b) Now prove that ghg^{-1} and h have the same order without any computations by using Problem 9(c) and Example 3 from Lecture 15.

(c) Prove that gh and hg have the same order. **Hint:** Use (a) (or (b)).

Hint for (a): Let $n = o(h)$, so that $h^n = e$. First show that $(ghg^{-1})^n = e$ as well (if you do not see how to do this, start with $n = 2$, see the pattern, then generalize). Note that the equality $(ghg^{-1})^n = e$ DOES NOT mean that $o(ghg^{-1}) = n$. It only means that $o(ghg^{-1}) \leq n = o(h)$ as there could exist $m < n$ such that $(ghg^{-1})^m = e$ as well. Show that the latter is impossible by contradicting the assumption $n = o(h)$.

Problem 11: Let D_8 be the octic group (the group of isometries of a square – for the definition see online Lecture 10 or the end of Chapter 7 in Pinter;

note that Pinter calls this group D_4) and Q_8 the quaternion group (see the definition on wikipedia).

(a) Find the order of each element in both D_8 and Q_8.
(b) Prove that D_8 and Q_8 are not isomorphic. **Hint:** Use Problem 9(c).