Homework #2. Due on Thursday, February 4th, in class

Reading:

1. For this assignment: Online lectures 2 (ordered rings part) and 3 and [Gilbert, §2.1,2.2].
2. For next week's classes: Online lectures 4 and 5, [Gilbert, §2.3,2.4] and [Pinter, §22]

Problems:

Problem 1: Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

Problem 2: Consider the following "proof" by induction: For each $n \in \mathbb{N}$ let $P(n)$ be the statement

$$\sum_{i=0}^{n} 2^i = 2^{n+1}. \qquad (***)$$

Claim: $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: "$P(n-1) \Rightarrow P(n)$." Assume that $P(n-1)$ is true for some $n \in \mathbb{N}$. Then $\sum_{i=0}^{n-1} 2^i = 2^n$. Adding 2^n to both sides, we get $\sum_{i=0}^{n-1} 2^i + 2^n = 2^n + 2^n$, whence $\sum_{i=0}^{n} 2^i = 2^{n+1}$, which is precisely $P(n)$. Thus, $P(n)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all n. □

(a) Show that the statement $P(n)$ is false (it is actually false for any n).
(b) Explain why the above "proof" does not contradict the principle of mathematical induction, that is, find a mistake in the above "proof" (Hint: the mistake is in the general logic).

Problem 3: In online lecture 3 it is proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$. Moreover, it is shown that such a_n and b_n satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.
(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1 + \sqrt{3})^n = c_n + d_n\sqrt{3}$.
(c) (bonus) Find a simple formula relating c_n and d_n (similar to the one in (a)) and prove it.

Problem 4: Given $n, k \in \mathbb{Z}$ with $0 \le k \le n$, define the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(recall that $0! = 1$).

(a) Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for any $1 \le k < n$ (direct computation).

(b) Now prove the binomial theorem: for every $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n.$$

Hint: Use induction on n. For the induction step write $(a+b)^{n+1} = (a+b)^n \cdot (a+b)$ and use part (a).

Note: In Problems 5(a) and 6(a) below you are allowed to make an extra assumption that R is a commutative ring with 1 (this does not make the proof considerably easier, but makes it possible to quote several previously established results).

Problem 5:

(a) Let R be an ordered ring. Prove that $x^2 > 0$ for every nonzero $x \in R$. **Hint:** Consider two cases.

(b) Use (a) to prove that $\mathbb{C}$ (complex numbers) is not an ordered ring (no matter how we try to define the set of positive elements).

Problem 6:

(a) Let R be an ordered ring. Prove that if $xy = 0$ for some $x, y \in R$, then $x = 0$ or $y = 0$. **Hint:** prove this by contrapositive.

(b) Let R be any ring which satisfies the conclusion of (a) ($xy = 0 \Rightarrow x = 0$ or $y = 0$ for all $x, y \in R$). Prove that multiplicative cancellation law holds in R, that is, if $xz = yz$ for some $x, y, z \in R$, then $x = y$ or $z = 0$.

Problem 7: Let R be a finite ring (that is, a ring, with finitely many elements), and suppose that $|R| > 1$. Prove that R cannot be an ordered ring. There is a hint on the next page, but first try to solve it without looking at the hint.

Hint for 7: Assume by way of contradiction that R is ordered; since $|R| > 1$, R must have at least one nonzero element, hence by axiom (O1) there must be at least one $x \in R$ such that $x > 0$. Now use Problem 3(a) from HW #1 repeatedly and transitivity of $>$ to reach a contradiction.