Homework #12. Not due

For this assignment: online notes (Lectures 24-26) and class notes (Lectures 24-25)

Problems:

Problem 1: Let $\mathbb{Z}[i]$ be the set of all complex numbers of the form $a+bi$ with $a,b\in\mathbb{Z}$. Prove that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. This ring is called **Gaussian integers.**

Problem 2: Let $S=\{a+b\sqrt{2}+c\sqrt{3} : a,b,c\in\mathbb{Z}\}$.

(a) Let T be a subring of $\mathbb{R}$ which contains 1 and $\sqrt{2}$ and $\sqrt{3}$. Prove that T contains all elements of S.

(b) Prove that S is NOT a subring of $\mathbb{R}$.

(c) Find the minimal subring of $\mathbb{R}$ which contains all elements of S. First guess what the answer should be, call your answer S_1 (step 1), then prove that S_1 is a subring (step 2), and finally prove that S_1 is the minimal subring containg S (step 3).

Problem 3: Let R be a commutative ring with 1 and I an ideal of R.

(a) Suppose that $1\in I$. Prove that $I=R$.

(b) Suppose that R is a field. Prove that $I=R$ or $I=\{0\}$. **Hint:** Reduce (b) to (a).

Problem 4: Let $R=\mathbb{Z}[x]$ (polynomials with coefficients in $\mathbb{Z}$), and let

$$I=\{a_0+a_1x+\ldots+a_nx^n : \text{ each } a_i\in\mathbb{Z} \text{ and } a_0 \text{ is even. }\}$$

(a) Prove that I is an ideal of R.

(b) Prove that I is a non-principal ideal, that is, $I\neq fR$ for any $f\in R$. **Hint:** Consider three cases.
  (i) f is a non-constant polynomial
  (ii) f is an even constant
  (iii) f is an odd constant.

Problem 5: Use FTH for rings to prove that $\mathbb{R}[x]/(x^2+1)\mathbb{R}[x]\cong\mathbb{C}$ (make sure to read the entire online Lecture 26 before working on this problem).