Homework #11. Due on Thursday, April 28th

Reading:

1. For this assignment: Lecture 22-23, [Pinter, §15,16] and [Gilbert, §4.6].
2. For next weeks classes: Lectures 24-26, [Pinter, §17-19]

Problems:

Note: Recall that at the beginning of Lecture 22 we gave a different description of the octic group D_8: if we denote by r the rotation by 90 degrees (in any direction) and by s any of the 4 reflections in D_8, then $D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ as a set; moreover, the multiplication table of D_8 is uniquely determined by the relations $r^4 = s^2 = e$ and $rs = sr^3$.

The correspondence with the original notations for the elements of D_8 (introduced in Lecture 10) is as follows: if we set $r = r_1$ and $s = s_1$, then $r_0 = e, r_1 = r, r_2 = r^2, r_3 = r^3, s_1 = s, s_2 = sr, s_3 = sr^2$ and $s_4 = sr^3$.

In Problems 1 and 5 of this assignment which deal with D_8 you can use either the old or the new notations, but please be consistent. Problems are formulated using the new notations (which are probably more convenient for computational purposes).

Problem 1: Let $G = D_8$, the octic group, and $H = \langle r^2 \rangle = \{e, r^2\}$. Describe the elements of the quotient group G/H and compute the multiplication table for G/H. Show details of your computation (some sample computations were done in Lecture 22 on April 19th). Make sure that in the multiplication table you do not use multiple names for the same element of G/H.

Problem 2: Let $G = (\mathbb{Z}_{12}, +)$ and $H = \langle [4] \rangle$, the cyclic subgroup generated by [4].

(a) Describe the elements of the quotient group G/H and compute the "multiplication" table for G/H (the word "multiplication" is in quotes because the group operation in G is addition).

(b) Deduce from your computation in (a) that G/H is isomorphic to $\mathbb{Z}_4$.

(c) Now give a different proof of the isomorphism $G/H \cong \mathbb{Z}_4$ using FTH.

Problem 3: Let A and B be a groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\}$ be the subset of G consisting of all elements whose second component is identity. Use FTH to prove that $\widetilde{A}$ is a normal subgroup of G and the quotient group $G/\widetilde{A}$ is isomorphic to B.

Problem 4: This problem deals with the group $\mathbb{Q}/\mathbb{Z}$, the quotient of the group $(\mathbb{Q}, +)$ of rationals with addition by the subgroup of integers.

(a) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

(b) Find all elements of order 12 in $\mathbb{Q}/\mathbb{Z}$ and prove your answer.

Warning: Since elements of quotient groups are defined as cosets, it is common to misinterpret the notion of the order for such element as the size (cardinality) of the corresponding coset. This is NOT the right interpretation. By the order here we mean the usual notion of the order of group elements (the minimal n such that ...).

Problem 5: Before doing this problem read the full subsection on transversals in the online version of Section 23 (only a brief part of it was discussed in class).

In each of the following examples, find a transversal of H in G. Also decide whether there exists a transversal which is a subgroup: if yes, exhibit such a transversal; if not, prove why.

(a) $G = \mathbb{Z}_6$, $H = \langle [2] \rangle$.

(b) $G = \mathbb{Z}_9$, $H = \langle [3] \rangle$.

(c) $G = D_8$, $H = \langle r \rangle = \{e, r, r^2, r^3\}$, the rotation subgroup.

(d) $G = D_8$, $H = \langle r^2 \rangle = \{e, r^2\}$.

Problem 6: The goal of this problem is to establish a simple relation between centralizers and conjugacy classes: let G be a finite group, $x \in G$, let $C(x)$ be the centralizer of x and $K(x)$ the conjugacy class of x. Then

$$|K(x)| = \frac{|G|}{|C(x)|} \qquad (***)$$

(a) Let $g_1, g_2 \in G$. Prove that $g_1 x g_1^{-1} = g_2 x g_2^{-1} \iff g_1 C(x) = g_2 C(x)$. **Hint:** Use Theorem 19.2.

(b) Now use (a) to show that $|K(x)| = [G : C(x)]$, the index of $C(x)$ in G.

Since $[G : C(x)] = \frac{|G|}{C(x)}$ by the (proof of) Lagrange Theorem, (b) implies formula (***).

Problem 7: Let G be the group from Problem 5 in HW#10. Give an alternative solution to part (b) of that problem as follows:

(i) first show (by direct computation) that for every non-identity element $x \in G$ either $|C(x)| = |F|$ or $|C(x)| = |F| - 1$.

(ii) use (i) and Problem 6 to show that $|K(x)| = |F|$ or $|K(x)| = |F| - 1$ for every non-identity element $x \in G$.