Homework #10. Due Thursday, April 21st

Reading:

1. For this assignment: Lectures 19-21, [Pinter, §8,13,14] and [Gilbert, §4.4, 4.5]. Make sure to read about even and odd permutations which we have not discussed in class (see second half of [Pinter, §8] and a brief online summary)
2. For next week's classes: Lecture 22-23, [Pinter, §15,16] and [Gilbert, §4.6].

Problems:

Problem 1: Let G be a group and H a subgroup of G. Consider the following relation $\sim$ on G:

$$g \sim k \iff g^{-1}k \in H.$$

(i) Prove that $\sim$ is an equivalence relation.

(ii) Prove that for every $g \in G$ its equivalence class with respect to $\sim$ is equal to the left coset gH.

Problem 2: Let G be a group and H a subgroup of G. In each of the following examples describe left cosets of H (in G). Find the number of distinct cosets and list all elements in each coset.

(a) $G = \mathbb{Z}_{12}$, $H = \langle [3] \rangle$.

(b) $G = D_8$ (the octic group), $H = \{r_0, r_1, r_2, r_3\}$ (the rotation subgroup).

(c) $G = D_8$, $H = \langle s_1 \rangle = \{r_0, s_1\}$ (recall that s_1 is the reflection wrt $y = 0$).

For (b) and (c) state the answer using the notations introduced in Lecture 10.

Problem 3: Let G be a group and H a subgroup of G.

(a) Let $g \in G$. Prove that $gH = H$ if and only if $g \in H$. (**Hint:** This is not hard to prove directly, but the result follows easily from Theorem 19.2 or from Problem 1(b)). State the analogous result for right cosets.

(b) Suppose that H has index 2 in G. Prove that H is normal in G (you will likely need (a) for your proof). **Note:** Usually, to prove that a subgroup is normal, conjugation criterion (Theorem 20.2) is easier to use than definition, but this problem is a rare exception. **Hint:** see the end of the assignment.

Problem 4: Let G be any group, consider $G \times G$, the direct product of two copies of G, and let $H = \{(g, g) \in G \times G\}$, that is, H is the set of all elements of $G \times G$ for which the first component is equal to the second component.

(a) Prove that H is a subgroup of $G \times G$. It is common to call this H the *diagonal subgroup* of $G \times G$. Note that if $G = \mathbb{R}$ (with addition) and we identify $G \times G$ with the plane $\mathbb{R}^2$, then H is the "diagonal" line $x = y$.

(b) Now use the conjugation criterion to prove that H is a normal subgroup of $G \times G$ if and only if G is abelian.

Problem 5: Let F be a finite field, and let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in F \text{ and } a \neq 0 \right\}$.

(a) (practice) Prove that G is a subgroup of $GL_2(F)$, so that G itself is a group (with matrix multiplication)

(b) Let x be any non-identity element of G, and let $K(x)$ be the conjugacy class of x. Prove that $|K(x)| = |F|$ or $|K(x)| = |F| - 1$. You can solve this problem by a (more or less) direct computation.

Note: The only examples of finite fields we have seen so far are $\mathbb{Z}_p$ where p is prime. There are more complicated finite fields, but it is not hard to describe all of them (up to isomorphism).

Problem 6: Before doing this problem read about even and odd permutations in Pinter and in the online notes.

(a) Write the permutation $(1, 2)(3, 4, 5)(6, 7, 8, 9)(10, 11, 12)(13, 14)$ as a product of transpositions.

(b) Let $f \in S_n$ be a cycle of length k. Prove that f is even if k is odd, and f is odd if k is even.

(c) Let $f \in S_n$. Write f as a product of disjoint cycles $f = f_1 f_2 \dots f_r$, and let k_i be the length of f_i for each i. Suppose that the "length sequence" $\{k_1, k_2, \dots, k_r\}$ contains a even numbers and b odd numbers. For instance, the length sequence of the permutation in part (a) is $\{2, 3, 4, 3, 2\}$, so $a = 3$ and $b = 2$.

Among the following 4 statements exactly one is correct. Find the correct statement and prove it.

(i) f is even if and only if a is even

(ii) f is even if and only if a is odd

(iii) f is even if and only if b is even

(iv) f is even if and only if b is odd

Problem 7:

(a) Consider the permutations $g = (1, 3, 5)(2, 4, 7, 8)$ and $f = (1, 7, 5, 6)(2, 8, 9)(3, 4)$ in S_9. Compute gfg^{-1} (you should be able to write down the answer right away).

(b) Consider the permutations $f = (1, 4, 6)(2, 3, 5)$ and $h = (3, 4, 6)(1, 5, 7)$ in S_7. Find $g \in S_7$ such that $gfg^{-1} = h$, $g(1) = 1$ and $g(3) = 3$.

(c) Let $f = (1, 2, 3)$ considered as an element of S_6, and let $C(f)$ be the centralizer of f in S_6 (recall that centralizers were defined in HW#6). Prove that $|C(f)| = 18$. **Hint:** Use the conjugation formula.

Hint for Problem 3: Since H has index 2 in G, there are only two left cosets, one of which is H itself – use this to describe the other coset. Then do the same for right cosets. Now recall that we need to prove $xH = Hx$ for every $x \in G$. Consider two cases: $x \in H$ and $x \notin H$.