

A. EVEN AND ODD PERMUTATIONS (BRIEF SUMMARY)

Recall that a **transposition** is a cycle of length 2.

Lemma A.1. *Any permutation $f \in S_n$ can be written as a product of transpositions.*

Proof. Since any permutation can be written as a product of disjoint cycles, it is sufficient to write each cycle as a product of transpositions. The latter can be done using the following formula, which is verified by direct computation:

$$(i_1, i_2, \dots, i_k) = (i_1, i_k)(i_1, i_{k-1}) \dots (i_1, i_2). \quad (\text{T})$$

□

Definition. A permutation $f \in S_n$ is called even (respectively, odd) if f can be written as a product of an even (respectively, odd) number of transpositions.

Note that there are several ways to write a given $f \in S_n$ as a product of transpositions. For instance, let $f = (3, 5, 4)(1, 2, 3, 4) \in S_5$. Then applying formula (T) above to each of the cycles $(3, 5, 4)$ and $(1, 2, 3, 4)$, we get $f = (3, 4)(3, 5)(1, 4)(1, 3)(1, 2)$, so f is a product of 5 transpositions.

On other hand, we can first write f as a product of disjoint cycles (in this example there will be just one cycle) and then use (T). This gives $f = (3, 5, 4)(1, 2, 3, 4) = (1, 2, 5, 4) = (1, 4)(1, 5)(1, 2)$, so f is a product of 3 transpositions.

This non-uniqueness could cause a problem with the definition of even/odd permutations, that is, it could theoretically happen that some permutation is both even or odd (which would make terminology very confusing). Fortunately, this problem does not occur thanks to the following theorem.

Theorem A.2. *Suppose that $f \in S_n$ can be written as a product of k transpositions and also as a product of l transpositions for some $k, l \in \mathbb{Z}_{>0}$. Then k and l are both even or both odd.*

Proof. See [GG, Theorem 4.3]. Another proof is outlined in the Bonus problem from Homework#10. □

The next proposition shows how to determine whether the product of two permutations is even or odd.

Proposition A.3. *Let $f, g \in S_n$.*

- (i) *If f and g are both even or both odd, then fg is even.*
- (ii) *If f is even and g is odd, or if f is odd and g is even, then fg is odd.*

Proof. This is because

- (a) If f is a product of k transpositions and g is a product of l transpositions, then fg is a product of $k + l$ transpositions and
- (b) The integer $k + l$ is even \iff k and l are both even or both odd.

□

Definition. The set of all even permutations in S_n is denoted by A_n .

Corollary A.4. *The following hold:*

- (1) *A_n is a subgroup of S_n .*
- (2) *The index $[S_n : A_n] = 2$ (assuming $n \geq 2$). This is equivalent to saying that exactly half of all permutations in S_n are even.*

Proof. (1) This follows fairly easily from Proposition A.3 and is left as an exercise.

(2) will be proved later using FTH (fundamental theorem of homomorphisms). □

Terminology: The groups A_n are called the alternating groups.

Exercises:

1. Write $f = (1, 3, 7)(5, 6)(2, 4, 8, 9)$ as a product of transpositions.
2. List all elements of A_4 .