

### Assessment exam

This exam will test your familiarity with basics of logic, basic properties of sets and functions as well as basic proof techniques (contradiction and induction). The goal of this exam is to help you decide whether you should take Math 3000 Transition to Higher Mathematics prior to/concurrently with Math 3354. A specific recommendation regarding Math 3000 will be made based on the results of the exam.

**0.** Did you already take one of the following classes: Math 3000 Transition to Higher Mathematics, Math 3310 Basic Real Analysis, CS 2102 Discrete Mathematics? If yes, list the classes and when you took them.

**1.** Prove by contradiction that  $\sqrt{12}$  is irrational directly from the definition of an irrational number (in particular, do not take for granted that  $\sqrt{3}$  is irrational).

**2.** Define the sequence  $\{a_n\}$  by  $a_1 = 4$  and  $a_n = 2a_{n-1} - 3$  for all  $n \geq 2$ . Use mathematical induction to prove that  $a_n = 2^{n-1} + 3$  for all  $n$ .

**3.** Consider the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined by the formulas  $f(x) = x^2$  for all  $x \in \mathbb{R}$  and  $g(y) = y^2$  for all  $y \in \mathbb{R}$ . Is it true that  $f = g$  as functions? Justify your answer.

**4.** Suppose somebody defines certain property of integers called *exciting*. You do not know what it means for an integer to be exciting; all you know is that every integer is either exciting or not exciting (but not both). Consider the following three statements:

- (a) Every exciting integer is even
- (b) Every even integer is exciting
- (c) Every odd integer is not exciting.

Two of the statements (a),(b) and (c) are logically equivalent (that is, you can deduce that if one of them is true, then the other must be true as well, without knowing what exciting means). Determine which two statements are logically equivalent and explain why.

**5.** For each of the following functions determine whether it is injective (one-to-one) and whether it is surjective (onto). Here  $\mathbb{N}$  stands for natural numbers (=positive integers),  $\mathbb{R}$  for real numbers and  $\mathbb{R}_{\geq 0}$  for non-negative real numbers. Give a detailed proof.

- (a)  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = 2x$
- (b)  $f : \mathbb{N} \rightarrow \mathbb{N}$  given by  $f(x) = 2x$

(c)  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x^2$

(d)  $f : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  given by  $f(x) = x^2$

**6.** Let  $f : X \rightarrow Y$  be a function. Given a subset  $A$  of  $X$ , let

$$f(A) = \{f(x) : x \in A\}$$

be the image of  $A$  under  $f$  (this is a subset of  $Y$ ). Given a subset  $B$  of  $Y$ , let

$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

be the preimage (=inverse image) of  $B$  under  $f$ , that is,  $f^{-1}(B)$  is the set of all elements of  $X$  which get mapped to an element of  $B$  under  $f$ .

For each of the following statements determine whether it is true (for all possible functions) or false (for at least one function). If the statement is true, prove it; if it is false, give a specific counterexample.

(i) If  $B$  is a subset of  $Y$ , then  $f(f^{-1}(B)) \subseteq B$

(ii) If  $B$  is a subset of  $Y$ , then  $f(f^{-1}(B)) \supseteq B$

(iii) If  $A$  and  $C$  are subsets of  $X$ , then  $f(A \cap C) = f(A) \cap f(C)$

(iv) If  $B$  and  $D$  are subsets of  $Y$ , then  $f^{-1}(B \cap D) = f^{-1}(B) \cap f^{-1}(D)$