20. Normal subgroups

20.1. **Definition and basic examples.** Recall from last time that if G is a group, H a subgroup of G and $g \in G$ some fixed element the set $gH = \{gh : h \in H\}$ is called a left coset of H.

Similarly, the set $Hg = \{hg : h \in H\}$ is called a right coset of H.

Definition. A subgroup H of a group G is called normal if $gH = Hg$ for all $g \in G$.

The main motivation for this definition comes from quotient groups which will be discussed in a couple of weeks.

Let us now see some examples of normal and non-normal subgroups.

Example 1. *Let G be an abelian group. Then any subgroup of G is normal.*

Example 2. *Let G be any group. Recall that the center of G is the set*

$$Z(G) = \{x \in G : gx = xg \text{ for all } g \in G\}.$$

By Homework#6.3, $Z(G)$ is a subgroup of G. Clearly, $Z(G)$ is always a normal subgroup of G; moreover, any subgroup of $Z(G)$ is normal in G.

Example 3. $G = S_3$, $H = \langle(1,2,3)\rangle = \{e, (1,2,3), (1,3,2)\}$.

Let $g = (1,2)$. Then

$$gH = \{(1,2), (1,2)(1,2,3), (1,2)(1,3,2)\} = \{(1,2), (2,3), (1,3)\}$$
$$Hg = \{(1,2), (1,2,3)(1,2), (1,3,2)(1,2)\} = \{(1,2), (1,3), (2,3)\}.$$

Note that while there exists $h \in H$ s.t. $gh \neq hg$, we still have $gH = Hg$ as sets.

The above computation does not yet prove that H is normal in G since we only verified $gH = Hg$ for a single g. To prove normality we would need to do the same for all $g \in G$. However, there is an elegant way to prove normality in this example, given by the following proposition.

Proposition 20.1. *Let G be a group and H a subgroup of index 2 in G. Then H is normal in G.*

Proof. This will be one of the problems in Homework#10. □

Recall from Lecture 19 that the index of H in G, denoted by $[G : H]$, is the number of left cosets of H in G and that if G is finite, then $[G : H] = \frac{|G|}{|H|}$. In

Example 3 we have $|G| = 6$ and $|H| = 3$, so $[G : H] = 2$ and Proposition 20.1 can be applied.

Finally, we give an example of a non-normal subgroup:

Example 4. $G = S_3$, $H = \langle(1,2)\rangle = \{e, (1,2)\}$.

To prove this subgroup is not normal it suffices to find a single $g \in G$ such that $gH \neq Hg$. We will show that $g = (1,3)$ has this property.

We have $gH = \{(1,3), (1,3)(1,2)\} = \{(1,3), (1,2,3)\}$ and $Hg = \{(1,3), (1,2)(1,3)\} = \{(1,3), (1,3,2)\}$. Since $\{(1,3), (1,2,3)\} \neq \{(1,3), (1,3,2)\}$ (as sets), H is not normal.

20.2. **Conjugation criterion of normality.**

Definition. Let G be a group and fix $g, x \in G$. The element gxg^{-1} is called the conjugate of x by g.

Theorem 20.2 (Conjugation criterion)**.** *Let G be a group and H a subgroup of G. Then H is normal in G $\iff$ for all $h \in H$ and $g \in G$ we have $ghg^{-1} \in H$. In other words, H is normal in G $\iff$ for every element of H, all conjugates of that element also lie in H.*

Proof. "$\Rightarrow$" Suppose that H is normal in G, so for every element $g \in G$ we have $gH = Hg$. Hence for every $h \in H$ we have $gh \in gH = Hg$, so $gh = h'g$ for some $h' \in H$. Multiplying both sides on the right by g^{-1}, we get $ghg^{-1} \in H$. Thus, we showed that $ghg^{-1} \in H$ for all $g \in G, h \in H$, as desired.

"$\Leftarrow$" Suppose now for all $g \in G, h \in H$ we have $ghg^{-1} \in H$. This means that $ghg^{-1} = h'$ for some $h' \in H$ (depending on g and h). The equality $ghg^{-1} = h'$ can be rewritten as $gh = h'g$. Since $h'g \in Hg$ by definition, we get that $gh \in Hg$ for all $h \in H, g \in G$, so $gH \subseteq Hg$ for all $g \in G$.

Since the last inclusion holds for all $g \in G$, it will remain true if we replace g by g^{-1}. Thus, $g^{-1}H \subseteq Hg^{-1}$ for all $g \in G$. Using Lemma 19.1 (associativity of multiplication of subsets in a group), multiplying the last inclusion by g on both left and right, we get $Hg \subseteq gH$.

Thus, for all $g \in G$ we have $gH \subseteq Hg$ and $Hg \subseteq gH$, and therefore $gH = Hg$. □

20.3. **Applications of the conjugation criterion.**

Theorem 20.3. *Let G and G' be groups and $\varphi : G \to G'$ a homomorphism. Then $\operatorname{Ker}(\varphi)$ is a normal subgroup of G.*

Proof. Let $H = \mathrm{Ker}\,(\varphi)$. We already know from Lecture 16 that H is a subgroup of G, so it suffices to check normality. We will do this using the conjugation criterion.

So, take any $h \in H$ and $g \in G$. By definition of the kernel we have $\varphi(h) = e'$ (the identity element of G'). Hence $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)e'\varphi(g)^{-1} = e'$, so $ghg^{-1} \in \mathrm{Ker}\,(\varphi) = H$. Therefore, H is normal by Theorem 20.2. □

Here are two more examples of application of the conjugation criterion

Example 5. *Let A and B be any groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\} \subseteq G$, the set of elements of G whose second component is the identity element of B.*

It is not hard to show that $\widetilde{A}$ is a subgroup of G and $\widetilde{A} \cong A$ (one can think of $\widetilde{A}$ as a canonical copy of A in G).

We claim that $\widetilde{A}$ is normal in G. Indeed, take any $g \in G$ and $h \in A$. Thus, $g = (x, y)$ and $h = (a, e_B)$ for some $a, x \in A$ and $y \in B$. Then $g^{-1} = (x^{-1}, y^{-1})$, so $ghg^{-1} = (x, y)(a, e_B)(x^{-1}, y^{-1}) = (xax^{-1}, ye_By^{-1}) = (xax^{-1}, e_B) \in \widetilde{A}$. Thus, $\widetilde{A}$ is normal by Theorem 20.2.

Example 6. *Let F be a field. Let*

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in F, ac \neq 0 \right\} \quad \textit{and} \quad H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$$

In Lecture 12 we proved that G is a subgroup of $GL_2(F)$ (so G itself is a group). We also know that H is a subgroup $GL_2(F)$ (by Homework #7.5); since clearly $H \subseteq G$, it follows that H is a subgroup of G.

Using conjugation criterion, it is not difficult to check that H is normal in G.