Homework #5. Due Thursday, February 26th, in class

Reading:

1. For this assignment: Sections 3.1, 3.2 + class notes (Lectures 10-11).
2. For next week's classes: 3.3 and part of 3.4 (before Theorem 3.24).

Problems:

Required reading 1: Read the online version of Lecture 11 before doing this assignment (some of this material was not discussed in class).

Required reading 2: Read pp. 155-158 of the book (Section 3.2) before class on Tuesday and prepare questions about unclear points.

Problem 1: In each of the following examples determine whether the given set G is a group with respect to a given operation. If G is a group, prove why (that is, verify all the axioms); if G is not a group, state at least one axiom which does not hold and explain why.

(i) $G = (\mathbb{R} \setminus \mathbb{Q}, +)$, the set of all irrational numbers with addition
(ii) $G = (\mathbb{Q}_{>0}, \cdot)$, the set of all rational numbers with multiplication

Problem 2: Let $G = \mathbb{R} \setminus \{-1\}$ be the set of real numbers different from -1, and define the binary operation $*$ on G by $x * y = x + y + xy$. Prove that $(G, *)$ is a group, find its identity element and explicit formula for the inverse of x. **Warning:** None of the four axioms in this example is obvious.

Problem 3: Let R be a ring with 1 (not necessarily commutative), and let $R^\times$ be the set of invertible elements of R, that is,

$$R^\times = \{a \in R : \text{ there exists } b \in R \text{ such that } ab = ba = 1\}.$$

Prove that $R^\times$ is closed with respect to multiplication (that is, if $x, y \in R^\times$, then $xy \in R^\times$). As mentioned in class, this is the main thing one needs to check to show that $R^\times$ is a group with respect to multiplication.

Problem 4: Compute the multiplication tables for the groups $\mathbb{Z}_5^\times, \mathbb{Z}_8^\times$ and $\mathbb{Z}_{10}^\times$ (here the superscript $\times$ has the same meaning as in Problem 3). Recall that invertible elements of $\mathbb{Z}_n$ are described in Theorem 9.1.

In Problems 5 and 6 below we use multiplicative notation in groups (note that this notation was not used in the proof of Theorem 11.1).

Problem 5: Let G be a group. Prove part (d) of Theorem 11.1 from class: if $az = e$ for some $a, z \in G$, then $a = z^{-1}$. Note: by definition, $a = z^{-1}$ if

$az = e$ AND $za = e$. What you have to show is that if $az = e$, then the other condition $za = e$ holds automatically.

Problem 6: Let G be a group.

(a) Prove that for any $a, b \in G$ the equation $ax = b$ has exactly one solution $x \in G$. Do the same for the equation $xa = b$.

(b) Deduce from (b) that every row and column of the multiplication table of G contains exactly one element of G (Sudoku puzzle property).

Problem 7: Let F be a field. Recall from Lecture 10 that $GL_2(F)$ denotes the set of all **invertible** 2×2 matrices with coefficients in F. The set $GL_2(F)$ is a group with respect to matrix multiplication (the the identity element of $GL_2(F)$ is the identity matrix, and the inverse of $A \in GL_2(F)$ is the inverse matrix in the usual sense). In order to determine whether a 2×2 matrix A lies in $GL_2(F)$ one can use the following result from linear algebra:

Theorem: *Let F be a field and let $n \geq 2$ be an integer. Then an $n \times n$ matrix $A \in Mat_n(F)$ is invertible if and only if* $\det(A) \neq 0$.

Also recall that the determinant of a 2×2 matrix is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Thus, $GL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \text{ and } ad - bc \neq 0. \right\}$

(a) Use Problem 5 to prove the following formula for inverses in $GL_2(F)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Recall that if $\lambda \in F$ is a scalar, then by definition $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

(b) Let $F = \mathbb{Z}_7$ and $A = \begin{pmatrix} [1] & [2] \\ [3] & [4] \end{pmatrix}$. Find A^{-1} (and simplify your answer). Answer the same question for $F = \mathbb{Z}_5$.

Problem 8: (practice) Section 3.2: 14 (page 151).

Problem 9: (practice) Let $G = \{r_0, r_1, r_2, r_3, s_1, s_2, s_3, s_4\}$ be the group of 8 xy-plane transformations preserving a square in $\mathbb{R}^2$. Recall that r_0 is the identity transformation, r_1, r_2, r_3 are counterclockwise rotations by 90, 180 and 270 degrees, respectively, and s_1, s_2, s_3, s_4 are reflections with respect to $y = 0$, $y = x$, $x = 0$ and $y = -x$, respectively.

(a) Prove that G is non-commutative.

(b) Let $H = \{r_0, r_2, s_1, s_3\}$ (so H is a subset of G containing 4 elements). Prove that H is a subgroup of G and compute the multiplication table of H. The definition of a subgroup is given in Section 3.3 and will be given in class on Tuesday, February 28. At this point proving that H is a subgroup will probably have to be done by computing all 16 products directly (but only 4 of those 16 computations will require work).