Homework #12. Not due

For this assignment: Sections 5.1, 6.1, 6.2 and class notes (Lectures 24-26)

Problems:

Note that problems 3 and 6(b) were done in class on Thu, April 23rd.

Problem 1: Let $\mathbb{Z}[i]$ be the set of all complex numbers of the form $a + bi$ with $a, b \in \mathbb{Z}$. Prove that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. This ring is called **Gaussian integers.**

Problem 2: Let $S = \{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in \mathbb{Z}\}$.

(a) Let T be a subring of $\mathbb{R}$ which contains 1 and $\sqrt{2}$ and $\sqrt{3}$. Prove that T contains all elements of S.

(b) Prove that S is NOT a subring of $\mathbb{R}$.

(c) Find the minimal subring of $\mathbb{R}$ which contains all elements of S. First guess what the answer should be, call your answer S_1 (step 1), then prove that S_1 is a subring (step 2), and finally prove that S_1 is the minimal subring containg S (step 3).

Problem 3: Let R be a commutative ring with 1 and I an ideal of R.

(a) Suppose that $1 \in I$. Prove that $I = R$.

(b) Suppose that R is a field. Prove that $I = R$ or $I = \{0\}$. **Hint:** Reduce (b) to (a).

Problem 4: Let R be a commutative ring with 1.

(a) Fix $a \in R$, and let $I = aR$, the principal ideal of R generated by a. Prove that I is the minimal ideal of R containing a.

(b) Now fix two elements $a, b \in R$, and let

$$I = aR + bR = \{x \in R : x = ar + bs \text{ for some } r, s \in R\}.$$

Prove that I is the minimal ideal of R containing a and b.

Hint: First prove that I is an ideal of R containing a and b and then show that if J is any ideal of R containing a and b, then J contains I.

Problem 5: Let $a, b \in \mathbb{Z}$, and let I be the minimal ideal of $\mathbb{Z}$ containing both a and b. Use Problem 4 and one of the problems from Homework#2 to prove that $I = d\mathbb{Z}$ where $d = gcd(a, b)$. State your argument clearly.

Remark: If $a_1, \ldots, a_k$ are elements of a ring R, the minimal ideal of R containing all these elements is commonly denoted by $(a_1, \ldots, a_k)$. With this notation, the result of Problem 5 maybe restated as $(a, b) = (d)$ where $d = gcd(a, b)$. This explains why the greatest common divisor of a and b is often denoted simply by (a, b).

Problem 6: Let $R = \mathbb{Z}[x]$ (polynomials with coefficients in $\mathbb{Z}$), and let

$$I = \{a_0 + a_1 x + \ldots + a_n x^n : \text{ each } a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even. } \}$$

(a) Use Problem 2 to prove that I is the minimal ideal of R containing 2 and x.

(b) Prove that I is a non-principal ideal, that is, $I \neq fR$ for any $f \in R$.
Hint: Consider three cases.
(i) f is a non-constant polynomial
(ii) f is an even constant
(iii) f is an odd constant.

Problem 7: Consider the quotient group $\mathbb{Q}/\mathbb{Z}$. Recall that the group operation on $\mathbb{Q}/\mathbb{Z}$ is denoted by $+$ and defined by

$$(a + \mathbb{Z}) + (b + \mathbb{Z}) = (a + b) + \mathbb{Z}.$$

Suppose now we want to turn $\mathbb{Q}/\mathbb{Z}$ into a ring and define multiplication on $\mathbb{Q}/\mathbb{Z}$ by

$$(a + \mathbb{Z}) \cdot (b + \mathbb{Z}) = ab + \mathbb{Z}.$$

Show that such multiplication will NOT be well defined.