Homework #11. Due on Thursday, April 23rd

Reading:

1. For this assignment: Sections 4.6 and 5.1 + online notes from Lectures 21-23.

2. For next weeks classes: online lectures 24-26 and sections 5.1, 6.1 and 6.2 of the book.

Problems:

Problem 1: Let $G = D_8$, the octic group, and $H = \{r_0, r_2\}$. Describe the elements of the quotient group G/H and compute the multiplication table for G/H. Show details of your computation (a sample computation will be done in class on April 14th). Make sure that in the multiplication table you do not use multiple names for the same element of G/H.

Problem 2: Let $G = (\mathbb{Z}_{12}, +)$ and $H = \langle [4] \rangle$, the cyclic subgroup generated by [4].

(a) Describe the elements of the quotient group G/H and compute the "multiplication" table for G/H (the word "multiplication" is in quotes because the group operation in G is addition).

(b) Deduce from your computation in (a) that G/H is isomorphic to $\mathbb{Z}_4$.

(c) Now give a different proof of the isomorphism $G/H \cong \mathbb{Z}_4$ using FTH.

Problem 3: Let A and B be a groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\}$ be the subset of G consisting of all elements whose second component is identity. Use FTH to prove that $\widetilde{A}$ is a normal subgroup of G and the quotient group $G/\widetilde{A}$ is isomorphic to B.

Problem 4: This problem deals with the group $\mathbb{Q}/\mathbb{Z}$, the quotient of the group $(\mathbb{Q}, +)$ of rationals with addition by the subgroup of integers.

(a) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

(b) Find all elements of order 12 in $\mathbb{Q}/\mathbb{Z}$ and prove your answer.

Warning: Since elements of quotient groups are defined as cosets, it is common to misinterpret the notion of the order for such element as the size (cardinality) of the corresponding coset. This is NOT the right interpretation. By the order here we mean the usual notion of the order of group elements (the minimal n such that ...).

Problem 5: Before doing this problem read the full subsection on transversals in the online version of Section 23 (only part of it was discussed in class).

In each of the following examples, find a transversal of H in G. Also decide whether there exists a transversal which is a subgroup: if yes, exhibit such a transversal; if not, prove why.

(a) $G = \mathbb{Z}_6$, $H = \langle [2] \rangle$.

(b) $G = \mathbb{Z}_9$, $H = \langle [3] \rangle$.

(c) $G = D_8$, $H = \{r_0, r_1, r_2, r_3\}$, the rotation subgroup.

(d) $G = D_8$, $H = \{r_0, r_2\}$. **Hint:** Use classification of subgroups of D_8.

Problem 6: The goal of this problem is to prove the following formula stated in Lecture 21: let G be a finite group, $x \in G$, let $C(x)$ be the centralizer of x and $K(x)$ the conjugacy class of x. Then $|K(x)| = \frac{|G|}{|C(x)|}$.

(a) Let $g_1, g_2 \in G$. Prove that $g_1 x g_1^{-1} = g_2 x g_2^{-1} \iff g_2^{-1} g_1 \in C(x) \iff g_2 C(x) = g_1 C(x)$ (the second equivalence easily follows from one of the problems in HW#10).

(b) Now use (a) and the Lagrange theorem (applied to $H = C(x)$) as stated in the book to prove that $|K(x)| = \frac{|G|}{|C(x)|}$.

Problem 7: For each of the following groups describe all conjugacy classes (find the number of conjugacy classes and explicitly list all elements in each class):

(a) $G = D_8$, the octic group. If you use notations for the elements of D_8 different from the ones introduced in class, state clearly what each symbol means.

(b) $G = Q_8$, the quaternion group.

You can solve this problem directly from definition of conjugacy classes, but you can save some time by using Problem 6.