Homework #10. Due Thursday, April 16th

Reading:

1. For this assignment: Sections 4.4, 4.5 and the second part of 4.1 (even/odd permutations and conjugacy classes in S_n), online notes on even/odd permutations + class notes (Lectures 19-21)

2. For next week's classes: read online Lecture 22 before Tuesday class and online Lecture 23 before Thursday class (this part is MANDATORY); also read Section 4.6 (quotient groups)

Problems:

Problem 1: Let G be a group and H a subgroup of G. Consider the following relation $\sim$ on G:

$$g \sim k \iff g^{-1}k \in H.$$

(i) Prove that $\sim$ is an equivalence relation.

(ii) Prove that for every $g \in G$ its equivalence class with respect to $\sim$ is equal to the left coset gH.

Problem 2: Let G be a group and H a subgroup of G. In each of the following examples describe left cosets of H (in G). Find the number of distinct cosets and list all elements in each coset.

(a) $G = \mathbb{Z}_{12}$, $H = \langle [3] \rangle$.

(b) $G = D_8$ (the octic group), $H = \{r_0, r_1, r_2, r_3\}$ (the rotation subgroup).

(c) $G = D_8$, $H = \langle s_1 \rangle = \{r_0, s_1\}$ (recall that s_1 is the reflection wrt $y = 0$).

Problem 3: Let G be a group and H a subgroup of G.

(a) Let $g \in G$. Prove that $gH = H$ if and only if $g \in H$. (**Hint:** This is not hard to prove directly, but the result follows easily from Theorem 19.2 or from Problem 1(b)). State the analogous result for right cosets.

(b) Suppose that H has index 2 in G. Prove that H is normal in G (you will likely need (a) for your proof). **Note:** Usually, to prove that a subgroup is normal, conjugation criterion (Theorem 20.2) is easier to use than definition, but this problem is a rare exception. **Hint:** see the end of the assignment.

Problem 4: Let $G = D_8$. For each subgroup of D_8, determine whether it is normal or not (for the complete list of subgroups of D_8 see solutions to homework#9). **Hint:** For subgroups which are normal, use one of the criteria discussed in class. Note that the center $Z(G) = \{r_0, r_2\}$ (verify this). For subgroups which are not normal, give a direct proof that they are not normal (using definition).

Problem 5: Let F be a field. Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in F \text{ and } ad \neq 0 \right\}$, and let $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$. Recall that we proved earlier in the course that B and U are subgroups of $GL_2(F)$ (and U is also a subgroup of B since clearly $U \subseteq B$).

(a) Use the conjugation criterion (Theorem 20.2) to prove that U is a normal subgroup of B.

(b) Prove that U is NOT a normal subgroup of $GL_2(F)$.

Note: You will probably need the formula for inverses in $GL_2(F)$ given in Homework#5.

Problem 6: Before doing this problem read about even and odd permutations either in the book (second part of 4.1) or in the online notes.

(a) Write the permutation $(1,2)(3,4,5)(6,7,8,9)(10,11,12)(13,14)$ as a product of transpositions.

(b) Let $f \in S_n$ be a cycle of length k. Prove that f is even if k is odd, and f is odd if k is even.

(c) Let $f \in S_n$. Write f as a product of disjoint cycles $f = f_1 f_2 \ldots f_r$, and let k_i be the length of f_i for each i. Suppose that the "length sequence" $\{k_1, k_2, \ldots, k_r\}$ contains a even numbers and b odd numbers. For instance, the length sequence of the permutation in part (a) is $\{2,3,4,3,2\}$, so $a = 3$ and $b = 2$.

Among the following 4 statements exactly one is correct. Find the correct statement and prove it.

(i) f is even if and only if a is even

(ii) f is even if and only if a is odd

(iii) f is even if and only if b is even

(iv) f is even if and only if b is odd

Problem 7:

(a) Consider the permutations $g = (1,3,5)(2,4,7,8)$ and $f = (1,7,5,6)(2,8,9)(3,4)$ in S_9. Compute gfg^{-1} (you should be able to write down the answer right away).

(b) Consider the permutations $f = (1,4,6)(2,3,5)$ and $h = (3,4,6)(1,5,7)$ in S_7. Find $g \in S_7$ such that $gfg^{-1} = h$, $g(1) = 1$ and $g(3) = 3$.

(c) Let $f = (1,2,3)$ considered as an element of S_6, and let $C(f)$ be the centralizer of f in S_6. Prove that $|C(f)| = 18$. **Hint:** Use the conjugation formula.

Bonus Problem: The goal of this problem is provide a different proof of the fact that the notion of even/odd permutation is well defined. Let $n \geq 2$ be an integer.

(a) For each $\sigma \in S_n$ let $P(\sigma) \in GL_n(\mathbb{Z})$ be the $n \times n$ matrix whose (i,j)-entry $P(\sigma)_{ij}$ is given by

$$P(\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j). \end{cases}$$

Prove that the map $P : S_n \to GL_n(\mathbb{Z})$ given by $\sigma \mapsto P(\sigma)$ is a homomorphism.

(b) Suppose that $\sigma \in S_n$ is a transposition. Prove that $\det P(\sigma) = -1$. **Hint:** The matrix $P(\sigma)$ is obtained from the identity matrix using a simple row operation.

(c) Deduce from (b) that if $\sigma \in S_n$ and σ is written as a product of transpositions in two different ways: $\sigma = \tau_1 \ldots \tau_k$ and $\sigma = \tau_1' \ldots \tau_l'$, then k and l are both even or both odd.

Hint for Problem 3: Since H has index 2 in G, there are only two left cosets, one of which is H itself – use this to describe the other coset. Then do the same for right cosets. Now recall that we need to prove $xH = Hx$ for every $x \in G$. Consider two cases: $x \in H$ and $x \notin H$.