A little memo on injective, surjective and bijective functions

1. Formally, a function is defined as follows (see [GG, §1.2]). Given two sets A and B, *a function* from A to B is a subset f of the Cartesian product $A \times B$ with the following property: for every $a \in A$ there exists unique $b \in B$ such that $(a, b) \in f$. If f is a function from A to B, we write $f : A \to B$.

Usually there is no need to treat functions in such a formal way, and you can think of a function $f : A \to B$ as a "rule" which assigns to each input value $a \in A$ the (uniquely determined) output value $f(a) \in B$. The key point is that by saying "$f : A \to B$ is a function" we assume that

(i) $f(a)$ is defined for ALL $a \in A$

(ii) $f(a) \in B$ for ALL $a \in A$

2. The set A in the above definition is called the *domain* of f, and the set B is called the *codomain* of f. Note that codomain is NOT the same as the range. The *range* of f denoted by $Range(f)$ or $f(A)$ is the set of all possible outputs of f:

$$Range(f) = f(A) = \{b \in B : b = f(a) \text{ for some } a \in A\}.$$

The range of f is always a subset of the codomain, but may be smaller than the codomain. For instance, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then the domain A is $\mathbb{R}$, the codomain B is also $\mathbb{R}$ while the range $f(A)$ is $\mathbb{R}_{\geq 0}$.

3. A function $f : A \to B$ is called *injective* if f sends (maps) different elements of A to different elements of B, that is,

$$\text{for any } a_1, a_2 \in A \text{ if } a_1 \neq a_2, \text{ then } f(a_1) \neq f(a_2).$$

Equivalently, f is injective if

$$\text{for any } a_1, a_2 \in A \text{ if } f(a_1) = f(a_2), \text{ then } a_1 = a_2.$$

4. A function $f : A \to B$ is called *surjective* if its range is equal to the codomain: $f(A) = B$. In other words, f is surjective if f hits every element of B. Yet another way to express surjectivity which is most convenient for practical verifications is the following:

- A function $f : A \to B$ is surjective if for every $b \in B$ the equation $f(x) = b$ can be solved for x, with $x \in A$.

Note that if a function $f : A \to B$ is not surjective, we can "force" it to become surjective by making the codomain smaller. Again, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Then f is not surjective. However, we can consider essentially the same function $\tilde{f}$ which is given by the same formula but has smaller codomain. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be given by $\tilde{f}(x) = x^2$. Then $\tilde{f}$ is surjective.

5. A function $f : A \to B$ is called *bijective* if f is both surjective and injective. Bijectivity of a function can also be verified using the following criterion:

Proposition. Let $f : A \to B$ be a function. The following are equivalent:

(i) f is bijective

(ii) f has an inverse function, that is, there exists a function $g : B \to A$ such that $g(f(a)) = a$ for all $a \in A$ and $f(g(b)) = b$ for all $b \in B$.

6. Potentially confusing terminology: As you know, injective functions are also called one-to-one functions. In Section 3.5 the book uses the terminology *one-to-one correspondence*. One-to-one correspondence is the same as a BIJECTIVE function, not an injective function.