Direct products and sums of groups.

1. Direct products. Let A and B be two groups. The direct product of A and B, denoted by $A \times B$, is defined as follows:
As a set, $A \times B$ is simply the direct product of the sets A and B, that is, $A \times B$ is the set of ordered pairs $\{(a, b) : a \in A, b \in B\}$. Note that if A and B are finite, then $|A \times B| = |A| \cdot |B|$.
The group operation on $A \times B$ is defined as follows: given $(a_1, b_1) \in A \times B$ and $(a_2, b_2) \in A \times B$, we set

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2) \qquad (***)$$

where $a_1 a_2$ is the product of a_1 and a_2 in A and $b_1 b_2$ is the product of b_1 and b_2 in B.
The identity element of $A \times B$ is the pair (e_A, e_B) where e_A is the identity element of A and e_B is the identity element of B. The inverse of an element (a, b) is (a^{-1}, b^{-1}). Verification of group axioms is straightforward.

2. Direct sums. Now suppose that A and B are abelian groups, and group operation in both A and B is denoted by $+$. Then it is common to talk about the direct sum of A and B instead of the direct product and denote it by $A \oplus B$. In this case the group operation on $A \oplus B$ is also denoted by $+$, and the formula (***) becomes

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

Since group operation in A and B is denoted by $+$, we denote identity elements of A and B by 0_A and 0_B, respectively, and we use additive notation for inverses ($-x$ instead of x^{-1}). Then the identity element of $A \oplus B$ is the pair $(0_A, 0_B)$, and the inverse of (a, b) is $(-a, -b)$.

3. Connection to vector spaces. Let $A = B = (\mathbb{R}, +)$, real numbers with respect to addition. Then $A \oplus B = \mathbb{R} \oplus \mathbb{R}$ is simply $\mathbb{R}^2$, the standard two-dimensional vector space over $\mathbb{R}$. The group operation on $\mathbb{R}^2$ is usual addition of vectors.

If A is any abelian group, $A \oplus A$ can be thought as vectors of length 2 whose components lie in A.

Finally, if A and B are arbitrary abelian groups, elements of $A \oplus B$ can be thought of as "mixed" vectors of length 2: *mixed* means that the first component lies in A, and the second component lies in B.