Homework #2. Due on Thursday, September 12th, 11:59pm on Canvas

Reading:

1. For this assignment: Online lectures 2 and 3. From Hungerford: Appendix C, 1.1 and 1.2.
2. For next week's classes: Online lectures 4, 5 and beginning of 6. From Hungerford: 1.2, 1.3 and the beginning of 2.1.

Online lectures are currently posted on the Spring 2016 webpage

`https://m-ershov.github.io/3354_Spring2016/`

Problems:

Problem 1: Given $n, k \in \mathbb{Z}$ with $0 \leq k \leq n$, define the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(recall that $0! = 1$).

(a) Prove that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for any $1 \leq k < n$ (direct computation).

(b) Now prove the binomial theorem: for every $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \ldots + \binom{n}{n-1} a b^{n-1} + \binom{n}{n} b^n.$$

Hint: Use induction on n. For the induction step write $(a+b)^{n+1} = (a+b)^n \cdot (a+b)$ and use part (a).

Note: In Problems 2 and 3 below we assume that R is an ordered ring (see definition in online Lecture 2) and $R_{>0}$ is its set of positive elements. Recall that if R is an ordered ring, given $x, y \in R$, we write $x > y$ if $x - y \in R_{>0}$. Also, by definition $x < y$ is the same as $y > x$, $x \geq y$ means $x > y$ or $x = y$ and $x \leq y$ means $x < y$ or $x = y$. In online Lecture 2 (Example 2.4) it is proved that $>$ is a transitive relation: if $x > y$ and $y > z$, then $x > z$.

Problem 2: Let R be an ordered ring. Prove the following basic properties of the relation $>$:

(a) ($<$ is anti-symmetric): there are no elements $x, y \in R$ such that $x < y$ and $y < x$.

(b) If $x > y$, then $x + z > y + z$ for all $z \in R$, that is, one can add a fixed element to both sides of an inequality. Recall that we argued in Lecture 1 that the corresponding property for equalites holds simply because $+$ is a binary operation (and does not use any axioms of addition). Explain why the situation is different for inequalities.

(c) If $x > y$ and $z > 0$, then $xz > yz$.

Problem 3:

(a) Let R be an ordered ring. Prove that $x^2 > 0$ for every nonzero $x \in R$. **Hint:** Consider two cases.

(b) Use (a) to prove that $\mathbb{C}$ (complex numbers) is not an ordered ring (no matter how we try to define the set of positive elements).

Problem 4:

(a) Use the induction axiom (called the Induction Property on page 4 of online Lecture 3) to prove (formally) that $n \geq 1$ for all $n \in \mathbb{Z}_{>0}$

(b) Let $x, y \in \mathbb{Z}$ with $x > y$. Use (a) to prove that $x \geq y + 1$.

Problem 5 (bonus): Deduce the well-ordering principle (axiom (Z3) on page 3 of online Lecture 3) from the induction axiom.

Hint: Let $Q(n)$ be some property of subsets of $\mathbb{Z}_{>0}$, and let $P(n)$ be the statement "Every subset of $\mathbb{Z}_{>0}$ satisfying $Q(n)$ has the smallest element." You are free to define $Q(n)$ in any way you like. Your goal is to define $Q(n)$ in such a way that

(i) Every non-empty subset of $\mathbb{Z}_{>0}$ satisfies $Q(n)$ for some n.

(ii) You can prove $P(1)$ directly.

(iii) You can prove the implication $P(n) \Rightarrow P(n+1)$ for all n.

Problem 6: Let $a, b, c \in \mathbb{Z}$ such that $c \mid a$ and $c \mid b$. Prove *directly from definition of divisibility* that $c \mid (ma + nb)$ for any $m, n \in \mathbb{Z}$ (do not refer to any divisibility properties proved in class).

Problem 7: Let $a, b, c \in \mathbb{Z}$ such that $c \mid ab$. Is it always true that $c \mid a$ or $c \mid b$? If the statement is true for all possible values of a, b, c, prove it; otherwise give a counterexample.