Homework #11. Due on Thursday, December 5th, 11:59pm on Canvas

Reading:

1. For this assignment: Online lectures 22, 23 and parts of 25, 26. From Hungerford: 7.5, 8.1 and 8.2.

2. For the remaining classes: Lecture 23 for the class on Monday, Nov 25, Lecture 25 for Monday, Dec 2 and Lecture 25 for Wednesday, Dec 4. From Hungerford: 8.4, 6.1 and 6.2 (in this order).

Online lectures are currently posted on the Spring 2016 webpage

`https://m-ershov.github.io/3354_Spring2016/`

Problems:

Problem 1: Let $G = D_8$, the octic group, and $H = \langle r^2 \rangle = \{e, r^2\}$. By HW#10, Problem 9, H is equal to $Z(G)$, the center of G, so it is normal and hence we can consider the quotient group G/H.

(a) Compute the multiplication table for G/H and show details of your computation (for a sample calculation see online Lecture 22). Make sure that in the multiplication table you do not use multiple names for the same element of G/H.

(b) Now prove that $G/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ without using your answer in (a). You may use the classification of groups of order 4.

Problem 2: Let $G = (\mathbb{Z}_{12}, +)$ and $H = \langle [4] \rangle$, the cyclic subgroup generated by $[4]$.

(a) Describe the elements of the quotient group G/H and compute the "multiplication" table for G/H (the word "multiplication" is in quotes because the group operation in G is addition).

(b) Deduce from your computation in (a) that G/H is isomorphic to $\mathbb{Z}_4$.

(c) Now give a different proof of the isomorphism $G/H \cong \mathbb{Z}_4$ using FTH.

Problem 3: Let A and B be a groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\}$ be the subset of G consisting of all elements whose second component is identity. Use FTH to prove that $\widetilde{A}$ is a normal subgroup of G and the quotient group $G/\widetilde{A}$ is isomorphic to B.

Problem 4: This problem deals with the group $\mathbb{R}/\mathbb{Z}$, the quotient of the group $(\mathbb{R}, +)$ of reals with addition by the subgroup of integers. Let $x \in \mathbb{R}$. Prove that $x + \mathbb{Z}$ (considered as an element of $\mathbb{R}/\mathbb{Z}$) has finite order if and only if $x \in \mathbb{Q}$.

Problem 5: Let R be a commutative ring with 1, and let I be an ideal of R. Prove that if I contains an element $r \in R$ which is invertible (in R), then $I = R$.

Problem 6: Let $R = \mathbb{Z}[x]$ (polynomials with coefficients in $\mathbb{Z}$), and let

$$I = \{a_0 + a_1 x + \ldots + a_n x^n : \text{ each } a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even. } \}$$

(a) Prove that I is an ideal of R

(b) Now prove that I is the minimal ideal of R containing 2 and x, that is, prove that any ideal containing 2 and x must contain I

(c) Prove that I is a non-principal ideal, that is, $I \neq fR$ for any $f \in R$.

Problem 7: Find all RING homomorphisms $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ (see the beginning of Lecture 26 for the definition of a ring homomorphism). **Hint:** If $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ is a ring homomorphism, then φ is also a group homomorphism where we consider $\mathbb{Z}_{10}$ as a group with addition. All group homomorphisms from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$ have been described in HW#9. Problem 4, and you only need to determine which of those homomorphisms are ring homomorphisms.

Problem 8: This problem deals with polynomials with coefficients in $\mathbb{Z}_3$. For $i = 0, 1, 2$ let $f(x) = x^2 + [i]_3 \in \mathbb{Z}_3[x]$, and let $R_i = \mathbb{Z}_3[x]/(f_i)$, the quotient of $\mathbb{Z}_3[x]$ by the principal ideal generated by f_i.

(a) Compute the multiplication table for the ring R_i. You do not have to write brackets in your answer.

(b) Prove that the rings R_1, R_2 and R_3 are pairwise non-isomorphic. **Hint:** For any two of these rings you can find a basic ring-theoretic property (which is preserved under homomorphisms) that holds for one of these rings and does not hold for the other.