Homework #1. Due on Friday, September 6th, 11:59pm on Canvas

Reading:

1. For this assignment: Online lectures: 1, the beginning of 2 (Example 2.1) and the beginning of 3 (subsection 3.1). From Hungerford: the beginning of Appendix B (pp. 509-512) and the beginning of Appendix C.
2. For next week's classes: Online lectures 3 and 4. I am planning to skip the content of Lecture 2 for now, but it would not hurt to go over it as well. From Hungerford: Appendix C, 1.1 and 1.2.

Online lectures are currently posted on the Spring 2016 webpage

`https://m-ershov.github.io/3354_Spring2016/`

Problems:

Problem 1: Let R be a commutative ring with 1. Prove the following equalities using only the ring axioms and results proved in class or online lectures.

(a) $-(xy) = (-x)y$ for all $x, y \in R$
(b) $(-1)(-1) = 1$
(c) $(-x)(-y) = xy$ for all $x, y \in R$
(d) $x(y - z) = xy - xz$ for all $x, y, z \in R$

Hint: Additive cancellation law (proved in lecture 1) can be used to solve many questions of this type as follows. Suppose that we want to prove inequality of the from $a = b$. By additive cancellation law, if we prove that $a + c = b + c$ for some $c \in R$, we can conclude that $a = b$. Note that the implication would work for any c, so c is for us to choose. The idea is to choose c in such a way that both expressions $a+c$ and $b+c$ can be simplified (using ring axioms) so that after simplification it becomes easy to prove that $a + c = b + c$.

Recall that by definition $x - y = x + (-y)$.

Problem 2: Let F be a field, and suppose that $xy = 0$ for some $x, y \in F$. Prove that $x = 0$ or $y = 0$. **Hint:** Consider two cases: $x = 0$ (in this case there is nothing to prove) and $x \neq 0$. Recall that in a field every nonzero element has multiplicative inverse.

Problem 3: Let X be any set, and let $R = \mathcal{P}(X)$ (the power set of X), that is, R is the of all subsets of X. As in online lecture 2, define addition $+$ and multiplication $\cdot$ on R by setting $A \cdot B = A \cap B$ (intersection) and

$A + B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ (symmetric difference ='exclusive or') for arbitrary $A, B \in R$ (that is, for arbitrary $A, B \subseteq X$). Prove that R with these operations is a commutative ring with 1.

Note: Multiplication axioms (M0)-(M3) are checked in online lecture 3, so you only need to check the addition axioms (A0)-(A4) and distributivity. You may want to read the beginning of Appendix B in Hungerford before doing this problem.

Hint: To check associativity of addition ($(A+B)+C = A+(B+C)$), take an arbitrary element $x \in X$, and consider 8 cases: case 1 ($x \in A$, $x \in B$, $x \in C$), case 2 ($x \in A$, $x \in B$, $x \notin C$) etc. In each case show that x belongs to both $(A+B)+C$ and $A+(B+C)$ or does not belong to either of those sets.

Problem 4: Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

Problem 5: Consider the following "proof" by induction: For each $n \in \mathbb{N}$ let $P(n)$ be the statement

$$\sum_{i=0}^{n} 2^i = 2^{n+1}. \qquad (***)$$

Claim: $P(n)$ is true for all $n \in \mathbb{N}$.

Proof: "$P(n-1) \Rightarrow P(n)$." Assume that $P(n-1)$ is true for some $n \in \mathbb{N}$. Then $\sum_{i=0}^{n-1} 2^i = 2^n$. Adding 2^n to both sides, we get $\sum_{i=0}^{n-1} 2^i + 2^n = 2^n + 2^n$, whence $\sum_{i=0}^{n} 2^i = 2^{n+1}$, which is precisely $P(n)$. Thus, $P(n)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all n. $\square$

(a) Show that the statement $P(n)$ is false (it is actually false for any n).
(b) Explain why the above "proof" does not contradict the principle of mathematical induction, that is, find a mistake in the above "proof" (Hint: the mistake is in the general logic).

Problem 6: In online lecture 3 it is proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$. Moreover, it is shown that such a_n and b_n satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1+\sqrt{3})^n = c_n + d_n\sqrt{3}$.

(c) (bonus) Find a simple formula relating c_n and d_n (similar to the one in (a)) and prove it.