Homework #9. Due on Thursday, November 3rd

Reading:

1. For this assignment: online lectures 17 and 18, Section 2.3 (pp. 74-80) and Section 3.2 (pp. 110-112)

2. For next week's classes: online lecture 19-20, Section 3.8 (pp. 164-168)

Problems:

Problem 1: (a) Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 7 & 2 & 6 & 4 \end{pmatrix}$ in two-line notation. Write f as in disjoint cycle form.

(b) Write the following element of S_9 as a product of disjoint cycles:

$$(1,2,4,6,7)(3,4,5,1,8)(9,2,3,5)$$

Problem 2: List all elements of S_3 in disjoint cycle form and compute the multiplication table of S_3.

Problem 3: Two elements f and g of S_n are said to have the same **cycle type** if their disjoint cycle forms contain the same number of cycles of each length. For instance, elements $(1,5,6)(2,3)(4,7)$ and $(1,7,8)(4,5)(3,6)$ of S_8 have the same cycle type. Show that elements of S_6 have 11 distinct cycle types. For each cycle type list one element of that type.

Problem 4: (a) Use the result of Problem 3 to determine possible orders of elements of S_6. Recall that if $f \in S_n$ is written as a product of disjoint cycles $f_1 f_2 \ldots f_r$ where f_1 has length k_1, ..., f_r has length k_r, then the order of f is the least common multiple of $k_1, k_2, \ldots, k_r$.

(b) Find the smallest $n \in \mathbb{N}$ for which S_n has an element of order 15 and prove your answer.

Problem 5: (a) Let $f, g \in S_n$ be two transpositions, that is, $f = (i,j)$ and $g = (k,l)$ for some i,j,k,l. What are the possible orders of the product fg? **Note:** By definition, a transposition is just a cycle of length 2. **Hint:** Consider three cases depending on the size of the set $\{i,j\} \cap \{k,l\}$ (note that $\{i,j\} \cap \{k,l\}$ is empty if and only f and g are disjoint cycles).

(b) Answer the same question when f is a transposition and g is a cycle of length 3.

Problem 6: Let G and H be finite groups such that $|G|$ and $|H|$ are coprime. Prove that any homomorphism $\varphi : G \to H$ must be trivial, that is, $\varphi(x) = e_H$ for all $x \in G$ where e_H is the identity element of H. **Hint:** Use the Range-Kernel theorem (see online Lecture 16) and Lagrange theorem (applied to a suitable subgroup). Note that this semester we are using the word ‘image’ instead of ‘range’ which is consistent with the terminology in the book (p. 51)

Problem 7: Let p and q be distinct primes, and let G be a group of order pq. Prove that one of the following two cases occurs:

(i) G is isomorphic to $\mathbb{Z}_{pq}$.

(ii) for every $x \in G$ either $x^p = e$ or $x^q = e$.

Problem 8: Use Lagrange theorem to prove Fermat’s little theorem: if p is prime, then $n^p \equiv n \mod p$ for any $n \in \mathbb{Z}$. **Hint:** Apply Corollary 18.1(B) to the group $\mathbb{Z}_p^\times = (\mathbb{Z}_p \setminus \{[0]\}, \cdot)$.

Problem 9: Before solving this problem, read the part of Lecture 17 dealing with Cayley’s theorem (we have not discussed this material in class). Let G be a finite group of order n, and let $\varphi : G \to S_n$ be an injective homomorphism from the proof of Cayley’s theorem:

(a) (practice) Describe φ explicitly (by computing $\varphi(g)$ for every $g \in G$) for each of the following groups: $G = \mathbb{Z}_4$, $G = S_3$ (in Lecture 18 we did the corresponding computation for $\mathbb{Z}_2 \times \mathbb{Z}_2$).

(b) (bonus) Prove that the following property holds for every group: if $g \in G$ and $m = o(g)$, then $\varphi(g)$ is a product of $\frac{n}{m}$ disjoint cycles of length m.

(c) (bonus) There are several different ways to “justify” the terminology “cyclic group”. Use the result of (b) to give one possible explanation of why cyclic groups are called cyclic.