Homework #8. Due Thursday, October 27th

Reading:

1. For this assignment: Online lectures 16 and 16A (pages 1-3), from the book: part of 3.3 discussing direct products (page 118), 3.4 and 3.7 up to Definition 3.7.5 (skip Example 3.7.8).

2. For next Tuesday's class: online lectures 16A (pages 1-3) and 17, Section 2.3 (pp. 74-80)

3. For next Thursday's class: online lecture 18, Section 3.2 (pp. 110-112)

Problems:

Problem 1:

(a) Let G be an abelian group and let m be an integer. Prove that the map $\varphi : G \to G$ given by $\varphi(x) = x^m$ is a homomorphism.

(b) Now use (a) and a theorem from class to solve Problem 2(a) in HW#6 without doing any computations.

Problem 2: Let G and H be groups and $\varphi : G \to H$ a homomorphism. For each of the following statements, determine whether it is true (in general) or false (in at least one case). If the statement is true, prove it; if it is false, give a specific counterexample.

(a) If H is abelian, then G is abelian

(b) If G is abelian, then H is abelian

(c) If G is abelian, then $\varphi(G)$ is abelian

(d) If G is abelian, then $\mathrm{Ker}\,(\varphi)$ is abelian

Problem 3: Let $G = (\mathbb{Z}_{12}, +)$. Define the map $\varphi : G \to G$ by $\varphi([x]) = 3[x] = [3x]$. Prove that φ is a homomorphism and compute its image and kernel. This problem is a warm-up for Problem 4.

Optional problem I: Let A and B be finite sets of the same cardinality, that is, $|A| = |B| = n < \infty$. Let $f : A \to B$ be a function. Prove that f is injective if and only if f is surjective.

Problem 4: Fix integers $n > 1$ and $m \in \mathbb{Z}$, and let $G = (\mathbb{Z}_n, +)$. Define the mapping $\varphi_m : G \to G$ by

$$\varphi_m([x]) = m[x] = [mx] \text{ for every } [x] \in \mathbb{Z}_n.$$

(a) Prove that $\varphi_m : G \to G$ is always a homomorphism. **Hint:** you already proved it in this homework.

(b) Prove that $\varphi_m(G)$ is equal to $\langle [m] \rangle$, the cyclic subgroup generated by $[m]$.

(c) Prove that φ_m is an isomorphism if and only if $gcd(m,n) = 1$. **Hint:** By part (a), the question is reduced to checking whether φ_m is bijective. By Optional Problem I it suffices to know when φ_m is surjective. To determine when φ_m is surjective, use (b) and one of the parts of Theorem 14.1.

(d) Now let ψ be an arbitrary homomorphism from G to G, Prove that $\psi = \varphi_m$ for some m with $0 \leq m \leq n-1$. **Hint:** Let $0 \leq m \leq n-1$ be such that $\psi([1]) = [m]$. Use the fact that ψ preserves group operation (addition in this case) to show that $\psi([x]) = \varphi_m([x])$ for any $x \in \mathbb{Z}$.

Problem 5: (practice) Let $m, n > 1$ be positive integer. For each integer x we denote by $[x]_n \in \mathbb{Z}_n$ the congruence class of x in $\mathbb{Z}_n$ and by $[x]_m \in \mathbb{Z}_m$ the congruence class of x in $\mathbb{Z}_m$. Now try to define a map $\varphi : \mathbb{Z}_n \to \mathbb{Z}_m$ by

$$\varphi([x]_n) = [x]_m.$$

(a) Prove that φ is a homomorphism whenever it is well defined.

(b) Now prove that φ is well defined $\iff$ $m \mid n$. **Hint:** By definition, φ is well defined if and only if the following implication holds for all $x, y \in \mathbb{Z}$:

$$\text{if } [x]_n = [y]_n, \text{ then } [x]_m = [y]_m. \qquad (***)$$

Thus, to prove (b) you need to show the following:

(i) If $m \mid n$, then (***) holds for all $x, y \in \mathbb{Z}$

(ii) If $m \nmid n$, then there exist $x, y \in \mathbb{Z}$ for which (***) does not hold.

(c) Find an injective homomorphism $\varphi : \mathbb{Z}_5 \to \mathbb{Z}_{10}$ (note that φ from (b) would not work as it will not be well defined).

Problem 6: Let G and H be groups.

(a) Prove that the direct product $G \times H$ is abelian if and only if G and H are both abelian.

(b) Let $\widetilde{G} = \{(g, e_H) : g \in G\}$ be the subset of $G \times H$ consisting of all elements whose second component is e_H. Prove that $\widetilde{G}$ is a subgroup of $G \times H$ and that this subgroup is isomorphic to G.

Problem 7: Use Proposition 3.4.3 from the book to prove that any two of the following groups are not isomorphic to each other: $\mathbb{Z}_{16}$, $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $D_8 \times \mathbb{Z}_2$. As usual D_8 is the octic group and $\mathbb{Z}_n$ is considered as a group with addition. Do not use Theorem 16.A.5 from online notes.

Problem 8: Let $n_1, \ldots, n_k$ be integers with $n_i \geq 2$ for all i, let $l = LCM(n_1, \ldots, n_k)$ and let $G = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$. As in Lecture 16A, we are using additive notation in G.

(a) Prove that $lg = 0$ for all $g \in G$ (so that the order of every element of G divides l).
(b) Now find $g \in G$ with $o(g) = l$ (and prove that g has this property)
(c) Deduce from (a) and (b) that G is cyclic if and only if the integers $n_1, \ldots, n_k$ are pairwise coprime. (This result is stated in Lecture 16A as Theorem 16.3).

Bonus problem:

(a) Let G be a group and let $\mathrm{Aut}\,(G)$ be the set of all automorphisms of G (= isomorphisms from G to G). Prove that elements of $\mathrm{Aut}\,(G)$ form a group with respect to composition. This group is called the *automorphism group of G*. **Hint:** This follows from Problem 5 of HW#7. What is the identity element of $\mathrm{Aut}\,(G)$?
(b) Let $G = \mathbb{Z}_n$ (with addition). Use the result of Problem 4 to prove that $\mathrm{Aut}\,(G)$ is isomorphic to $\mathbb{Z}_n^{\times}$ (with multiplication). **Hint:** This problem is much easier than it seems. Elements of $\mathrm{Aut}\,(G)$ are explicitly described in Problem 4. Use it to find a natural bijective mapping between $\mathrm{Aut}\,(G)$ and $\mathbb{Z}_n^{\times}$; then show that your mapping is in fact an isomorphism.