Homework #7. Due Thursday, October 20th

Reading:

1. For this assignment: Online lectures 13-15, Section 3.5 (up to the middle of page 138) and Section 3.4.
2. For next week's classes: Online lectures 15,16 and 16(a), up to Theorem 16.A.3. From the book: part of 3.3 discussing direct products (page 118), 3.4 and 3.7 up to Definition 3.7.5 (skip Example 3.7.8).

Problems:

Problem 1: Let x be an element of a group G, let $n = o(x)$, and assume that $n < \infty$. Let d be a positive divisor of n. Prove directly from definition of the order that $o(x^d) = \frac{n}{d}$.

Warning: to prove that an element y has order m it is not sufficient to check that $y^m = e$; you also need to show that m is the minimal positive integer with this property; equivalently, you also need to show that $y^k \neq e$ when $1 \leq k \leq m-1$.

Problem 2: (practice) Theorem 14.1 is applicable to any finite cyclic group G and any generator x of G. If $G = (\mathbb{Z}_n, +)$ for some n, we can use $x = [1]$ as a generator, in which case all assertions of the Theorem can be restated directly in terms of n. For instance, part (i) would say: "Every subgroup of $\mathbb{Z}_n$ is cyclic and is equal to $\langle [d] \rangle$ where d is a positive divisor of n". Restate other parts of Theorem 14.1 in a similar way.

Problem 3:

(a) Use the restatement of Theorem 14.1 from Problem 2 to do find all generators of $(\mathbb{Z}_{12}, +)$ and $(\mathbb{Z}_{15}, +)$

(b) Find all subgroups of $(\mathbb{Z}_{24}, +)$ and $(\mathbb{Z}_{36}, +)$ and draw the subgroup diagrams for these groups (see page 137 in § 3.5 for the definition of a subgroup diagram).

Problem 4: In Homework#6 it was shown that the group $G = \mathbb{Z}_9^\times$ is cyclic with generators [2] and [5] (and no other generators). Use Theorem 14.1 to show that once we found [2] to be a generator of G, we can deduce that [5] is also a generator and that there are no other generators without explicitly computing any other cyclic subgroups.

Problem 5: (practice) Prove that the relation $\cong$ of "being isomorphic" is an equivalence relation (Claim 15.1 from online Lecture 15).

Hint: To prove that $\cong$ is symmetric, show that if $\varphi : G \to G'$ is an isomorphism, then the inverse map $\varphi^{-1} : G' \to G$ is also an isomorphism. Since the inverse of a bijection is a bijection, you only need to show that $\varphi^{-1}(uv) = \varphi^{-1}(u)\varphi^{-1}(v)$ for all $u, v \in G'$. To prove this, take any $u, v \in G'$, and let $x = \varphi^{-1}(u), y = \varphi^{-1}(v)$. Then $\varphi(x) = u$ and $\varphi(y) = v$; at this point you can use the fact that φ is a isomorphism.

Problem 6:

(a) Let $G = (\mathbb{Z}_6, +)$ and $G' = (\mathbb{Z}_7^\times, \cdot)$. Prove that $G' \cong G$ and find an explicit isomorphism $\varphi : G \to G'$.

(b) (practice) Use map φ from (a) to show (explicitly) that the multiplication tables of G and G' can be obtained from each other by relabeling of elements.

Problem 7: Let F be a field and $G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\}$. As shown in HW#6, G is a subgroup of $GL_2(F)$ (and thus G is itself a group with respect to matrix multiplication). Find an isomorphism φ from $(\mathbb{R}, +)$ to G (and prove that φ is an isomorphism).

Problem 8: Let G be a group from Problem 1 in Homework #6: $G = \mathbb{R} \setminus \{-1\}$ as a set, and the operation $*$ on G is defined by $x * y = xy + x + y$. Prove that $(G, *)$ is isomorphic to $(\mathbb{R}\setminus\{0\}, \cdot)$ and find an explicit isomorphism between those groups.

Problem 9: Let $\varphi : G \to G'$ be an isomorphism, and let $g \in G$.

(a) Prove by induction that $\varphi(g^n) = \varphi(g)^n$ for every $n \in \mathbb{N}$.

(b) Prove that if $n \in \mathbb{N}$, then $g^n = e_G$ if and only if $\varphi(g)^n = e_{G'}$ (where e_G is the identity element of G and $e_{G'}$ is the identity element of G'). **Hint:** Use (a) and the fact that an isomorphism must send identity element to identity element (this will be proved in class next week).

(c) Use (b) to prove that $o(g) = o(\varphi(g))$ (Proposition 15.3 from online notes). Thus isomorphisms preserve orders of elements.

Problem 10: Let G be a group and $g, h \in G$.

(a) Prove that the elements ghg^{-1} and h have the same order by direct computation.

(b) Now prove that ghg^{-1} and h have the same order without any computations by using Problem 9(c) and Example 3 from Lecture 15.

(c) Prove that gh and hg have the same order. **Hint:** Use (a) (or (b)).

Hint for (a): Let $n = o(h)$, so that $h^n = e$. Start by showing that $(ghg^{-1})^n = e$ as well (if you do not see how to do this, start with $n = 2$, see the pattern, then generalize). As explained in the warning to #1, this is not the end of the problem.