Homework #5. Due Thursday, October 6th, in class

Reading:

1. For this assignment: Section 3.1, beginning of Section 2.3 (pages 71-73) + online Lectures 10-11.

2. For next week's class: beginning of Section 3.2 (up to the end of page 107) and online Lecture 12.

Problems:

Problem 1: In each of the following examples determine whether the given set G is a group with respect to a given operation. If G is a group, prove why (that is, verify all the axioms); if G is not a group, state at least one axiom which does not hold and explain why.

(i) $G = (\mathbb{R} \setminus \mathbb{Q}, +)$, the set of all irrational numbers with addition

(ii) $G = (\mathbb{Q}_{>0}, \cdot)$, the set of all POSITIVE rational numbers with multiplication

Problem 2: Let $G = \mathbb{R} \setminus \{-1\}$ be the set of real numbers different from -1, and define the binary operation $*$ on G by $x * y = x + y + xy$. Prove that $(G, *)$ is a group, find its identity element and explicit formula for the inverse of x. **Warning:** None of the four axioms in this example is obvious.

Problem 3: Let R be a ring with 1 (not necessarily commutative), and let $R^{\times}$ be the set of invertible elements of R, that is,

$$R^{\times} = \{a \in R : \text{ there exists } b \in R \text{ such that } ab = ba = 1\}.$$

Prove that $R^{\times}$ is closed with respect to multiplication (that is, if $x, y \in R^{\times}$, then $xy \in R^{\times}$). As mentioned in class, this is the main thing one needs to check to show that $R^{\times}$ is a group with respect to multiplication.

Problem 4: Compute the multiplication tables for the groups $\mathbb{Z}_7^{\times}, \mathbb{Z}_8^{\times}$ and $\mathbb{Z}_{10}^{\times}$ (here the superscript $\times$ has the same meaning as in Problem 3).

In Problems 5 and 6 below we use multiplicative notation in groups (note that this notation was NOT used in the proof of Theorem 11.1).

Problem 5: Let G be a group.

(a) Prove that for any $a, b \in G$ the equation $ax = b$ has exactly one solution $x \in G$. **Hint:** Consider the following two implications:

(i) If $ax = b$, multiplying both sides by a^{-1} on the left, we get $a^{-1}(ax) = a^{-1}b$, hence by (G1) and (G2) we have $x = a^{-1}b$.

(ii) If $x = a^{-1}b$, then by (G1) and (G2) $ax = a(a^{-1}b) = (aa^{-1})b = eb = b$.

One of these implications proves that the equation $ax = b$ has at least one solution and the other proves that the equation $ax = b$ has at most one solution. Decide which is which and explain why.

(b) Now do the same for the equation $xa = b$.

(c) Deduce from (a) and (b) that every row and column of the multiplication table of G contains exactly one element of G (Sudoku puzzle property).

Problem 6: A group G is called *abelian* (=commutative) if $xy = yx$ for ALL $x, y \in G$. Prove that a group G is abelian $\iff$ $(xy)^2 = x^2y^2$ for all $x, y \in G$.

Note/warning: By definition $g^2 = g * g$ where $*$ is the group operation. To prove that a group G is abelian, you need to show that $xy = yx$ for ALL $x, y \in G$ (you cannot pick x and y that you like).

Problem 7: Let F be a field. Recall from Lecture 10 that $GL_2(F)$ denotes the set of all **invertible** 2×2 matrices with coefficients in F. The set $GL_2(F)$ is a group with respect to matrix multiplication (the the identity element of $GL_2(F)$ is the identity matrix, and the inverse of $A \in GL_2(F)$ is the inverse matrix in the usual sense). In order to determine whether a 2×2 matrix A lies in $GL_2(F)$ one can use the following result from linear algebra:

Theorem: *Let F be a field and let $n \geq 2$ be an integer. Then an $n \times n$ matrix $A \in Mat_n(F)$ is invertible if and only if* $\det(A) \neq 0$.

Also recall that the determinant of a 2×2 matrix is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Thus, $GL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \text{ and } ad - bc \neq 0. \right\}$

(a) Prove the following formula for inverses in $GL_2(F)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Recall that if $\lambda \in F$ is a scalar, then by definition $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$ **Hint:** Computation will be very short if use a suitable part of Theorem 11.1.

(b) Let $F = \mathbb{Z}_7$ and $A = \begin{pmatrix} [1] & [2] \\ [3] & [4] \end{pmatrix}$. Find A^{-1} (and simplify your answer). Answer the same question for $F = \mathbb{Z}_5$.

Problem 8: Read about the octic group D_8 (Example 7 in Lecture 10) and compute its multiplication table. Use the notation for the elements of D_8 introduced in the above example $(r_0, r_1, r_2, r_3, s_1, s_2, s_3, s_4)$.