Homework #12. Not due in written form (but mandatory to complete)

Reading:

1. For this assignment: Lectures 25-26, Sections 5.1 (pp. 224-231), 5.2 and 5.3 (pp. 251-254).

2. For the last class (Dec 6): Integral domains (Sections 5.1, pp. 232-233) and Characteristic of a ring (Section 5.2, pp. 248-249).

Problems:

Problem 1: Let R be a commutative ring with 1, and let I be an ideal of R. Prove that if I contains an element $r \in R$ which is invertible (in R), then $I = R$.

Problem 2: Let R be a commutative ring with 1.

(a) Fix $a \in R$, and let $I = aR$, the principal ideal of R generated by a. Prove that I is the minimal ideal of R containing a.

(b) Now fix two elements $a, b \in R$, and let

$$I = aR + bR = \{x \in R : x = ar + bs \text{ for some } r, s \in R\}.$$

Prove that I is the minimal ideal of R containing a and b.

Hint: First prove that I is an ideal of R containing a and b and then show that if J is any ideal of R containing a and b, then J contains I.

Problem 3: Let $a, b \in \mathbb{Z}$, and let I be the minimal ideal of $\mathbb{Z}$ containing both a and b. Use Problem 2 and one of the problems from Homework#2 to prove that $I = d\mathbb{Z}$ where $d = gcd(a, b)$. State your argument clearly.

Remark: If $a_1, \ldots, a_k$ are elements of a ring R, the minimal ideal of R containing all these elements is commonly denoted by $(a_1, \ldots, a_k)$. With this notation, the result of Problem 5 maybe restated as $(a, b) = (d)$ where $d = gcd(a, b)$. This explains why the greatest common divisor of a and b is often denoted simply by (a, b).

Problem 4: Let $R = \mathbb{Z}[x]$ (polynomials with coefficients in $\mathbb{Z}$), and let

$$I = \{a_0 + a_1x + \ldots + a_nx^n : \text{ each } a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even. }\}$$

(a) Use Problem 2 to prove that I is the minimal ideal of R containing 2 and x.

(b) (done in class on Thu, Dec 1) Prove that I is a non-principal ideal, that is, $I \neq fR$ for any $f \in R$.

Problem 5: Find all RING homomorphisms $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$. **Hint:** If $\varphi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ is a ring homomorphism, then φ is also a group homomorphism where we consider $\mathbb{Z}_{10}$ as a group with addition. All group homomorphisms from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$ have been described in HW#8, and you only need to determine which of those homomorphisms are ring homomorphisms.

Problem 6: If R and S are rings, their direct product $R \times S$ is defined as the set of ordered pairs $\{(r, s) : r \in R, s \in S\}$ with addition an multiplication defined by $(r_1, s_1)+(r_2, s_2) = (r_1+r_2, s_1+s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2, s_1s_2)$. The zero element of $R \times S$ is the pair $(0_R, 0_S)$, and if both R and S are rings with 1, then the element $(1_R, 1_S)$ is the unity element of $R \times S$.
On Thu, Dec 1 in class we used FTH for rings to prove that

$$\mathbb{R}[x]/(x^2 + 1)\mathbb{R}[x] \cong \mathbb{C} \qquad (1).$$

Use analogous approach to prove that

$$\mathbb{R}[x]/(x^2 - 1)\mathbb{R}[x] \cong \mathbb{R} \times \mathbb{R} \qquad (2)$$

Hint: To prove (1) using FTH we defined $\varphi : \mathbb{R}[x] \to \mathbb{C}$ by $\varphi(f(x)) = f(i)$ and showed that $\mathrm{Ker}\,(\varphi) = (x^2 + 1)\mathbb{R}[x]$. The key reason for the latter equality is that $i^2 + 1 = 0$. To prove (2), define $\psi : \mathbb{R}[x] \to \mathbb{C}$ by

$$\psi(f(x)) = (f(a), f(b))$$

for suitably chosen $a, b \in \mathbb{R}$ and show that ψ has required properties.