Homework #11. Due on Thursday, December 1st

Reading:

1. For this assignment: Lectures 22-24, Sections 3.8 (pp. 168-175) and 5.1.
2. For the next three classes (Nov 22, 29 and Dec 1): Lectures 24-26, Sections 5.1, 5.2 and 5.3 (pp. 251-254)

Problems:

Note: Recall that at the beginning of Lecture 22 we gave a different description of the octic group D_8: if we denote by r the rotation by 90 degrees (in any direction) and by s any of the 4 reflections in D_8, then $D_8 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ as a set; moreover, the multiplication table of D_8 is uniquely determined by the relations $r^4 = s^2 = e$ and $rs = sr^3$.

The correspondence with the original notations for the elements of D_8 (introduced in Lecture 10) is as follows: if we set $r = r_1$ and $s = s_1$, then $r_0 = e, r_1 = r, r_2 = r^2, r_3 = r^3, s_1 = s, s_2 = sr, s_3 = sr^2$ and $s_4 = sr^3$.

In Problems 1 and 5 of this assignment which deal with D_8 you can use either the old or the new notations, but please be consistent. Problems are formulated using the new notations (which are probably more convenient for computational purposes).

Problem 1: Let $G = D_8$, the octic group, and $H = \langle r^2 \rangle = \{e, r^2\}$. Describe the elements of the quotient group G/H and compute the multiplication table for G/H. Show details of your computation (some sample computations were done in Lecture 22 on November 15th). Make sure that in the multiplication table you do not use multiple names for the same element of G/H.

Problem 2: Let $G = (\mathbb{Z}_{12}, +)$ and $H = \langle [4] \rangle$, the cyclic subgroup generated by [4].

(a) Describe the elements of the quotient group G/H and compute the "multiplication" table for G/H (the word "multiplication" is in quotes because the group operation in G is addition).

(b) Deduce from your computation in (a) that G/H is isomorphic to $\mathbb{Z}_4$.

(c) Now give a different proof of the isomorphism $G/H \cong \mathbb{Z}_4$ using FTH.

Problem 3: Let A and B be a groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\}$ be the subset of G consisting of all elements whose second component is identity. Use FTH to prove that $\widetilde{A}$ is a normal subgroup of G and the quotient group $G/\widetilde{A}$ is isomorphic to B.

Problem 4: This problem deals with the group $\mathbb{R}/\mathbb{Z}$, the quotient of the group $(\mathbb{R}, +)$ of reals with addition by the subgroup of integers. Let $x \in \mathbb{R}$. Prove that $x + \mathbb{Z}$ (considered as an element of $\mathbb{R}/\mathbb{Z}$) has finite order if and only if $x \in \mathbb{Q}$.

Problem 5: Before doing this problem read the full subsection on transversals in the online version of Section 23 (only a brief part of it was discussed in class).

In each of the following examples, find a transversal of H in G. Also decide whether there exists a transversal which is a subgroup: if yes, exhibit such a transversal; if not, prove why.

(a) $G = \mathbb{Z}_6$, $H = \langle [2] \rangle$.

(b) $G = \mathbb{Z}_9$, $H = \langle [3] \rangle$.

(c) $G = D_8$, $H = \langle r \rangle = \{e, r, r^2, r^3\}$, the rotation subgroup.

(d) $G = D_8$, $H = \langle r^2 \rangle = \{e, r^2\}$.

Problem 6: The goal of this problem is to establish a simple relation between centralizers and conjugacy classes: let G be a finite group, $x \in G$, let $C(x)$ be the centralizer of x and $K(x)$ the conjugacy class of x. Then

$$|K(x)| = \frac{|G|}{|C(x)|} \qquad (***)$$

(a) Let $g_1, g_2 \in G$. Prove that $g_1 x g_1^{-1} = g_2 x g_2^{-1} \iff g_1 C(x) = g_2 C(x)$. **Hint:** Use Theorem 19.2.

(b) Now use (a) to show that $|K(x)| = [G : C(x)]$, the index of $C(x)$ in G. **Hint:** find an explicit bijection between $K(x)$ and the quotient set $G/C(x)$.

Since $[G : C(x)] = \frac{|G|}{C(x)}$ by the (proof of) Lagrange Theorem, (b) implies formula (***).

Problem 7:

(a) Let $\mathbb{Z}[i]$ be the set of all complex numbers of the form $a + bi$ with $a, b \in \mathbb{Z}$. Prove that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. This ring is called **Gaussian integers.**

(b) (optional) Let $\mathbb{Q}[i]$ be the set of all complex numbers of the form $a + bi$ with $a, b \in \mathbb{Q}$. Prove that $\mathbb{Q}[i]$ is a subfield of $\mathbb{C}$. **Note:** If F is a field and S is a subset of F, to prove that S is a subfield you need to check that S is a subring and, in addition, S contains multiplicative inverses of all its nonzero elements (for any nonzero $s \in S$, the multiplicative inverse s^{-1} exists in F because F is a field, but you have to show that s^{-1} actually lies in S).

Problem 8: Let $S = \{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in \mathbb{Z}\}$.

(a) Let T be a subring of $\mathbb{R}$ which contains 1 and $\sqrt{2}$ and $\sqrt{3}$. Prove that T contains all elements of S.

(b) Prove that S is NOT a subring of $\mathbb{R}$.

(c) Find the minimal subring of $\mathbb{R}$ which contains all elements of S. First guess what the answer should be, call your answer S_1 (step 1), then prove that S_1 is a subring (step 2), and finally prove that S_1 is the minimal subring containing S (step 3).

Hint for (c): Your proof in part (b) should suggest which elements must be added to S to get a subring.

Bonus Problem: This is a continuation of the Bonus Problem from HW#10. We keep all the notations introduced in that problem.

Recall that we are trying to prove the following theorem.

Theorem: *Let p be an odd prime. Any group of order $2p$ is isomorphic to $\mathbb{Z}_{2p}$ or D_{2p} (the group of isometries of a regular p-gon).*

The following was established in HW#10 : Let G be a group of order $2p$, and assume that $G \not\cong \mathbb{Z}_{2p}$. Then G contains elements x, y such that

$$x^p = e, \quad y^2 = e, \quad yxy^{-1} = x^{-1} \qquad (***).$$

(i) Prove that $G = \{e, x, x^2, \ldots, x^{p-1}, y, yx, yx^2, \ldots, yx^{p-1}\}$

(ii) Now use (i) and the relations (***) to construct an explicit isomorphism from G to D_{2p} (and prove that your map is an isomorphism).