Homework #10. Due Thursday, November 17th

Reading:

1. For this assignment: online lectures 19-21, 17 (section 17.4) and notes on odd/even permutations (see spring 2016 webpage), sections 3.2 (pp. 81-83), 3.6 and 3.8 (pp. 164-167)

2. For next week's classes: online Lecture 22-23 and 3.8 (pp. 168-175)

Note: Several problems in this assignment deal with the octic group D_8. When working with this group, you should use either notations for its elements introduced in online Lecture 10 ($r_0, r_1, r_2, r_3, s_1, s_2, s_3, s_4$) or notations from the book introduced in Section 3.6 (words in symbols a and b), but be consistent (use the same notation in all problems). Note that in the book the octic group is denoted by D_4, not D_8. The correspondence between the notations in the book and Lecture 10 is given below:

$$r_0 = e,\ r_1 = a,\ r_2 = a^2,\ r_3 = a^3,\ s_3 = b,\ s_4 = ab,\ s_1 = a^2b,\ s_2 = a^3b.$$

Problems:

Problem 1: Let $\varphi : D_8 \to S_8$ be the homomorphism form the proof of Cayley's theorem. Describe φ explicitly (by computing $\varphi(g)$ for every $g \in D_8$) explicitly. You can use the version of the proof of Cayley's theorem from online Lecture 17 (in which case elements of $\varphi(D_8)$ will literally be elements of S_8) or the proof given on page 142 in the book (which is essentially the same as the proof given in class on Tue, Nov 8), in which case elements of $\varphi(D_8)$ will be permutations of the set $\{r_0, r_1, r_2, r_3, s_1, s_2, s_3, s_4\}$ (or $\{e, a, a^2, a^3, b, ab, ab^2, ab^3\}$), depending on the notations you use.

Problem 2: Let G be a group and H a subgroup of G. In each of the following examples describe left cosets of H (in G). Find the number of distinct cosets and list all elements in each coset.

(a) $G = \mathbb{Z}_{12}$, $H = \langle [3] \rangle$.

(b) $G = D_8$, $H = \{r_0, r_1, r_2, r_3\}$ (the rotation subgroup).

(c) $G = D_8$, $H = \langle s_1 \rangle = \{r_0, s_1\}$ (recall that s_1 is the reflection wrt $y = 0$).

Problem 3: Let G be a group and H a subgroup of G.

(a) Let $g \in G$. Prove that $gH = H$ if and only if $g \in H$. State the analogous result for right cosets.

(b) Suppose that H has index 2 in G. Prove that H is normal in G (you will likely need (a) for your proof). **Note:** Usually, to prove that a subgroup is normal, conjugation criterion (Theorem 20.2) is easier to use than definition, but this problem is a rare exception. **Hint:** see the end of the assignment.

Problem 4: Let $G = D_8$. For each subgroup of D_8, determine whether it is normal or not. The complete diagram of subgroups of D_8 can be found on page 148 of the book. **Hint:** For subgroups which are normal, use one of the criteria discussed in class. Note that the center $Z(G) = \{r_0, r_2\} = \{e, a^2\}$ (verify this). For subgroups which are not normal, give a direct proof that they are not normal (using definition).

Problem 5: Let G be a group, let H and K be subgroups of G, and suppose that H is normal in G. Let $HK = \{hk : h \in H, k \in K\}$. Prove that HK is a subgroup of G.

Problem 6: Before doing this problem read about even and odd permutations either in the book or in the class/online notes.

(a) Write the permutation $(1,2)(3,4,5)(6,7,8,9)(10,11,12)(13,14)$ as a product of transpositions.

(b) Let $f \in S_n$ be a cycle of length k. Prove that f is even if k is odd, and f is odd if k is even.

(c) Let $f \in S_n$. Write f as a product of disjoint cycles $f = f_1 f_2 \ldots f_r$, and let k_i be the length of f_i for each i. Suppose that the "length sequence" $\{k_1, k_2, \ldots, k_r\}$ contains a even numbers and b odd numbers. For instance, the length sequence of the permutation in part (a) is $\{2, 3, 4, 3, 2\}$, so $a = 3$ and $b = 2$.

Among the following 4 statements exactly one is correct. Find the correct statement and prove it.

(i) f is even if and only if a is even

(ii) f is even if and only if a is odd

(iii) f is even if and only if b is even

(iv) f is even if and only if b is odd

Problem 7: Before doing this problem, read the first 3 pages of online lecture 21.

(a) Consider the permutations $g = (1,3,5)(2,4,7,8)$ and $f = (1,7,5,6)(2,8,9)(3,4)$ in S_9. Compute gfg^{-1} (you should be able to write down the answer right away).

(b) Consider the permutations $f = (1,4,6)(2,3,5)$ and $h = (3,4,6)(1,5,7)$ in S_7. Find $g \in S_7$ such that $gfg^{-1} = h$, $g(1) = 1$ and $g(3) = 3$.

(c) Let $f = (1,2,3)$ considered as an element of S_6, and let $C(f)$ be the centralizer of f in S_6. Prove that $|C(f)| = 18$. **Hint:** Use the conjugation formula.

Bonus Problem: In Lecture 20 we briefly outlined the (start of the) proof of the following theorem which appears as Theorem 18.5 in the online notes.

Theorem: *Let p be an odd prime. Any group of order $2p$ is isomorphic to $\mathbb{Z}_{2p}$ or D_{2p} (the group of isometries of a regular p-gon).*

Here is a summary of what we said in class. Let G be a group of order $2p$. If G is cyclic, then $G \cong \mathbb{Z}_{2p}$ by Lecture 15, so assume from now on that G is NOT cyclic.

By Corollary 18.1(A) for every $x \in G$ we have $o(x)|2p$, so $o(x) = 1, 2, p$ or $2p$. Since we assume that G is not cyclic, $o(x)$ cannot equal $2p$, and of course $o(x) = 1$ if and only if $x = e$. Thus $o(x) = 2$ or p for every non-identity element $x \in G$.

Fact 1: G contains at least one element of order 2 and at least one element of order p.

Assuming Fact 1 without proof for the moment, let $x, y \in G$ be such that $o(x) = p$ and $o(y) = 2$, and let $H = \langle x \rangle$, the cyclic subgroup of G generated by x. Then $|H| = o(x) = p$. Thus the index $[G : H] = \frac{|G|}{|H} = \frac{2p}{p} = 2$, so H is normal in G. Since $x \in H$, by the conjugation criterion of normality we must have $yxy^{-1} \in H$, so

$$yxy^{-1} = x^k \text{ for some } 0 \le k \le p-1. \qquad (***)$$

Now the actual problem:

(a) Prove Fact 1. **Hint:** To prove the existence of an element of order 2 let $S = \{g \in G : g = g^{-1}\}$ and argue that $|S|$ is even. Then prove the existence of an element of order p by contradiction. Assume there is no such element. Then every non-identity element has order 2, so $g^2 = e$ for all $g \in G$. By HW#6.1 this implies that G is abelian. Now choose any two non-identity elements $a, b \in G$, and let H be the smallest subgroup of G containing them. Argue that $|H| = 4$ (using both commutativity of G and the identity $g^2 = e$) and reach a contradiction with Lagrange theorem.

(b) Prove that in the equation (***) above $k = 1$ or $k = p - 1$. **Hint:** Conjugate both sides of (***) by y and use the obtained equation and (***) again to show that $x^{k^2} = x$.

(c) Prove that if $k = 1$ (which means that x and y commute), the element xy has order $2p$ which contradicts the assumption that G is not cyclic.

In the remaining case $k = p - 1$ it is not hard to show that $G \cong D_{2p}$, but this part of the proof is postponed until the next assignment.

Hint for Problem 3: Since H has index 2 in G, there are only two left cosets, one of which is H itself – use this to describe the other coset. Then do the same for right cosets. Now recall that we need to prove $xH = Hx$ for every $x \in G$. Consider two cases: $x \in H$ and $x \notin H$.