**Homework #1. Due on Thursday, September 1st in class**

**Reading:**

1. For this assignment: Online lectures 1,2 and the beginning of 3.
2. For next week's classes: Online lectures 3,4 and 5 and Sections 1.1 and 1.2 of the book.

Online lectures are currently posted on last semester's webpage

`http://people.virginia.edu/~mve2x/3354_Spring2016`

**Problems:**

**Problem 1:** Let $R$ be a commutative ring with 1. Prove the following equalities using only the ring axioms and results proved in class or online lectures.

(a) $-(xy) = (-x)y$ for all $x, y \in R$
(b) $(-1)(-1) = 1$
(c) $(-x)(-y) = xy$ for all $x, y \in R$
(d) $x(y - z) = xy - xz$ for all $x, y, z \in R$

**Hint:** Additive cancellation law (proved in lecture 1) can be used to solve many questions of this type as follows. Suppose that we want to prove inequality of the from $a = b$. By additive cancellation law, if we prove that $a + c = b + c$ for some $c \in R$, we can conclude that $a = b$. Note that the implication would work for any $c$, so $c$ is for us to choose. The idea is to choose $c$ in such a way that both expressions $a+c$ and $b+c$ can be simplified (using ring axioms) so that after simplification it becomes easy to prove that $a + c = b + c$.

Recall that by definition $x - y = x + (-y)$.

**Problem 2:** Let $F$ be a field, and suppose that $xy = 0$ for some $x, y \in F$. Prove that $x = 0$ or $y = 0$. **Hint:** Consider two cases: $x = 0$ (in this case there is nothing to prove) and $x \neq 0$. Recall that in a field every nonzero element has multiplicative inverse.

**Note:** If $F$ was only assumed to be a commutative ring with unity, the above assertion would have been false in general. Can you think of an example?

**Problem 3:** Let $R$ be an ordered ring and $x, y, z \in R$. Prove that

(a) If $x > y$, then $x + z > y + z$
(b) If $x > y$ and $z > 0$, then $xz > yz$
(c) If $x > y$ and $z < 0$, then $xz < yz$

**Note:** You may use freely standard properties of ring operations (addition, subtraction and multiplication). However, all statement involving inequalities must be deduced directly from the axioms.

**Problem 4:** Prove by induction that the following equalities hold for any $n \in \mathbb{N}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$

**Problem 5:** Consider the following "proof" by induction: For each $n \in \mathbb{N}$ let $P(n)$ be the statement

$$\sum_{i=0}^{n} 2^i = 2^{n+1}. \qquad (***)$$

**Claim:** $P(n)$ is true for all $n \in \mathbb{N}$.

*Proof:* "$P(n-1) \Rightarrow P(n)$." Assume that $P(n-1)$ is true for some $n \in \mathbb{N}$. Then $\sum_{i=0}^{n-1} 2^i = 2^n$. Adding $2^n$ to both sides, we get $\sum_{i=0}^{n-1} 2^i + 2^n = 2^n + 2^n$, whence $\sum_{i=0}^{n} 2^i = 2^{n+1}$, which is precisely $P(n)$. Thus, $P(n)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n$. $\square$

(a) Show that the statement $P(n)$ is false (it is actually false for any $n$).

(b) Explain why the above "proof" does not contradict the principle of mathematical induction, that is, find a mistake in the above "proof" (Hint: the mistake is in the general logic).

**Problem 6:** In online lecture 3 it is proved that for every $n \in \mathbb{N}$ there exist $a_n, b_n \in \mathbb{Z}$ such that $(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}$. Moreover, it is shown that such $a_n$ and $b_n$ satisfy the following recursive relations: $a_1 = b_1 = 1$ and $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ for all $n \in \mathbb{N}$.

(a) Use the above recursive formulas and mathematical induction to prove that $a_n^2 - 2b_n^2 = (-1)^n$ for all $n \in \mathbb{N}$.

(b) Prove that for all $n \in \mathbb{N}$ there exist $c_n, d_n \in \mathbb{Z}$ such that $(1 + \sqrt{3})^n = c_n + d_n\sqrt{3}$.

(c) (bonus) Find a simple formula relating $c_n$ and $d_n$ (similar to the one in (a)) and prove it.