Homework #9. Due Thursday, November 4th

Reading:

1. For this homework assignment: Section 4.1 and the part of 4.4 dealing with the statement and applications of Lagrange theorem.

2. For Tuesday's class: Section 4.4 (cosets)

3. For Thursday's class: Section 4.5 (normal subgroups) and Theorem 4.24 on page 232.

Problems:

Problem 1: 4.1.1(f) and 4.1.2(f), pages 202-203.

Problem 2: Two elements f and g of S_n are said to have the same **cycle type** if their disjoint cycle forms contain the same number of cycles of each length. For instance, elements $(1,5,6)(2,3)(4,7)$ and $(1,7,8)(4,5)(3,6)$ of S_8 have the same cycle type. Show that elements of S_6 have 11 distinct cycle types. For each cycle type list one element of that type.

Problem 3: (a) Use the result of Problem 2 to determine possible orders of elements of S_6. Recall that if $f \in S_n$ is written as a product of disjoint cycles $f_1 f_2 \ldots f_r$ where f_1 has length k_1, ..., f_r has length k_r, then the order of f is the least common multiple of $k_1, k_2, \ldots, k_r$.

(b) Find the smallest $n \in \mathbb{Z}^+$ for which S_n has an element of order 15 and prove your answer.

Problem 4: (a) Let $f, g \in S_n$ be two transpositions, that is, $f = (i,j)$ and $g = (k,l)$ for some i,j,k,l. What are the possible orders of the product fg? **Note:** By definition, a transposition is just a cycle of length 2. **Hint:** Consider three cases depending on the size of the set $\{i,j\} \cap \{k,l\}$ (note that $\{i,j\} \cap \{k,l\}$ is empty if and only f and g are disjoint cycles).

(b) (optional) Answer the same question when f is a transposition and g is a cycle of length 3.

Problem 5: Let G be a group, and let H and K be subgroups of G. Suppose that

(i) H is contained in K,

(ii) $K \neq H$ and $K \neq G$

(iii) $|G| = 24$ and $|H| = 6$.

Determine $|K|$. State your proof clearly. **Hint:** Draw a picture.

Problem 6: Let G and H be finite groups such that $|G|$ and $|H|$ are relatively prime. Prove that any homomorphism $\varphi : G \to H$ must be trivial, that is, $\varphi(x) = e_H$ for any $x \in G$ where e_H is the identity element of H. **Hint:** Use the Range-Kernel theorem and Lagrange theorem (applied to a suitable subgroup).

Problem 7: Let p and q be distinct primes, and let G be a group of order pq. Prove that one of the following two cases occurs:

(i) G is isomorphic to $\mathbb{Z}_{pq}$.

(ii) for every $x \in G$ either $x^p = e$ or $x^q = e$.

Problem 8: Use Lagrange theorem to prove Fermat's little theorem: if p is prime, then $n^p \equiv n \mod p$ for any $n \in \mathbb{Z}$. **Hint:** Apply what we called Corollary B in class to the group $\mathbb{Z}_p^*$.

Bonus Problem: Describe all subgroups of the octic group D_8. **Hint:** First determine possible orders of subgroups. To describe all subgroups use the following fact proved in class: any group of order 4 is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (in particular, this implies that any group of order 4 is abelian). There will be a total of 10 subgroups (including the entire D_8 and the trivial subgroup).