Homework #7. Due Thursday, October 21st

Reading:

1. For this homework assignment: Sections 3.4 and 3.5.
2. For next week's classes: Section 3.6.

Problems:

Problem 1: Recall Theorem 14.1 stated in class:

Theorem 14.1. *Let G be a finite cyclic group, $n = |G|$ and x a generator of G. Let $k \in \mathbb{Z}$. The following hold:*

(i) $\langle x^k \rangle = \langle x^d \rangle$ *where* $d = gcd(n, k)$

(ii) $o(x^k) = n/gcd(n, k)$

(iii) x^k *is a generator of* $G \iff gcd(n, k) = 1$.

(iv) Every subgroup of G is cyclic and is equal to $\langle x^d \rangle$ where d is a positive divisor of n.

(v) If d and d' are distinct positive divisors of n, then $\langle x^d \rangle \neq \langle x^{d'} \rangle$

Part (i) was proved in class. The goal of this problem is prove the other parts. Note that parts (iii)-(v) are proved in the book, but in this problem you are asked to give slightly different proofs, following the outline given below.

First use the definition of the order to prove that if d is a positive divisor of n, then $o(x^d) = n/d$. Then use this fact, Corollary 13.2 (= Definition 3.18 from the book) and (i) to prove (ii).

Next prove the following lemma:

Lemma. *Let H be a finite group of order n and $y \in H$. Then y is a generator of $H \iff o(y) = n$.*

Then use this lemma and (ii) to prove (iii).

Next use Corollary 3.21 (from the book) and (i) to prove (iv). Finally, use (ii) and Corollary 13.2 to prove (v).

Problem 2: (a) Problem 3.4.9(b)(d). (b) Problem 3.4.10(b)(d)

Problem 3 *(practice):* 3.5.1 and 3.5.2. The main point of these problems is to show that the relation $\cong$ of "being isomorphic" is an equivalence relation on the collection of groups. 3.5.1 proves that $\cong$ is symmetric (if $G \cong G'$, then $G' \cong G$), and 3.5.2 proves that $\cong$ it transitive. The reflexivity of $\cong$ (every group is isomorphic to itself) is obvious. The equivalence classes with respect to this relation are called **isomorphism classes** of groups.

Hint: For Problem 3.5.1: you need to show that $\phi^{-1}(uv) = \phi^{-1}(u)\phi^{-1}(v)$ for any $u, v \in G'$. So, take any $u, v \in G'$, and let $x = \phi^{-1}(u), y = \phi^{-1}(v)$. Then $\phi(x) = u$ and $\phi(y) = v$; at this point you can use the fact that ϕ is a isomorphism.

Problem 4:

(a) Let $G = (\mathbb{Z}_6, +)$ and $G' = (\mathbb{Z}_7^*, \cdot)$. Prove that $G' \cong G$ and find an explicit isomorphism $\phi : G \to G'$. **Hint:** Use Example 3 from Lecture 14.

(b) (optional) Use map ϕ from (a) to show (explicitly) that the multiplication tables of G and G' can be obtained from each other by relabelling of elements.

Problem 5:

(a) Let $G = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ with matrix multiplication (note that G is a subgroup of $GL_2(\mathbb{R})$ by Problem 5 in Homework #6, so G itself is a group). Find an isomorphism ϕ from G to $(\mathbb{R}, +)$ (and prove that ϕ is an isomorphism).

(b) (bonus) $G = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \text{ and } a^2 + b^2 \neq 0 \right\}$ (again, G is a group by Problem 5 in Homework #6). Prove that G is isomorphic to $\mathbb{C}^*$ (nonzero complex numbers with multiplication).

Problem 6: Let G be a group, fix $g \in G$, and define a mapping $\phi : G \to G$ by $\phi(x) = g^{-1}xg$. Prove that ϕ is an isomorphism and find a formula for the inverse map $\phi^{-1} : G \to G$.

Problem 7: Let $\phi : G \to G'$ be an isomorphism, and let let $g \in G$.

(a) Prove by induction that $\phi(g^n) = \phi(g)^n$ for any $n \in \mathbb{Z}_{>0}$.

(b) Prove that if $n \in \mathbb{Z}_{>0}$, then $g^n = e_G$ if and only if $\phi(g)^n = e_{G'}$ (where e_G is the identity element of G and $e_{G'}$ is the identity element of G'). **Hint:** Use (a) and Theorem 3.26(a).

(c) Use (b) to prove that $o(g) = o(\phi(g))$. Thus isomorphisms preserve orders of elements.

Problem 8: Let G be a group and $g, h \in G$.

(a) Prove that the elements $g^{-1}hg$ and h have the same order by direct computation.

(b) Now prove that $g^{-1}hg$ and h have the same order without any computations by using Problems 6 and 7(c).

(c) Prove that gh and hg have the same order. **Hint:** Use (a) (or (b)).

Hint for (a): Let $n = o(h)$. First show that $(g^{-1}hg)^n = e$ as well (if you do not see how to do this, start with $n = 2$, see the pattern, then generalize). Note that the equality $(g^{-1}hg)^n = e$ DOES NOT mean that $o(g^{-1}hg) = n$. It only means that $o(g^{-1}hg) \leq n = o(h)$ as there could exist $m < n$ such that $(g^{-1}hg)^m = e$ as well. Show that the latter is impossible by contradicting the assumption $n = o(h)$.

Problem 9: Let D_8 be the octic group (the group of isometries of a square) and Q_8 the quarternion group defined in Exercise 3.1.28.

(a) Find the order of each element in both D_8 and Q_8.

(b) Prove that D_8 and Q_8 are not isomorphic. **Hint:** Use Problem 7(c).