Homework #5. Due Thursday, September 30th, in class

Reading:

For this homework assignment: Sections 2.6, 3.1 and 3.2.
Plan for next week's classes: 3.2 (finish), 3.3, 3.4 (start)

To hand in:

Problem 1: Section 3.1: 2, 4 (page 142). In both problems, if G is a group, prove why (that is, verify all the axioms); if G is not a group, state which axioms do not hold and explain why.

Problem 2: Let $n \in \mathbb{Z}^+$ and let $\mathbb{Z}_n^*$ be the set of all invertible elements in $\mathbb{Z}_n$, that is,

$$\mathbb{Z}_n^* = \{[x] \in \mathbb{Z}_n : \text{ there exists } [y] \in \mathbb{Z}_n \text{ such that } [x][y] = [y][x] = [1]\}.$$

Prove that $\mathbb{Z}_n^*$ is a group with respect to multiplication. **Hint:** The main thing to be checked is that $\mathbb{Z}_n^*$ is closed under multiplication (but make sure to explain why the remaining axioms hold as well). To prove that $\mathbb{Z}_n^*$ is closed under multiplication you may either

(i) try to prove that the set of invertible elements of any ring is closed under multiplication, that is, if u and v are invertible, then uv is invertible.

(ii) use characterization of invertible elements in $\mathbb{Z}_n$ given by Theorem 2.30 (page 109).

Problem 3: Section 3.1: 27 (page 143).

Problem 4: Let G be a group.

(a) Prove that if $az = e$ for some $a, z \in G$, then $a = z^{-1}$. Note: by definition, $a = z^{-1}$ if $az = e$ AND $za = e$. What you have to show is that if $az = e$, then the other condition $za = e$ holds automatically.

(b) Use (a) to prove the formula for the inverse of a product in a group: $(xy)^{-1} = y^{-1}x^{-1}$.

Problem 5: Let G be a group.

(a) Prove that for any $a, b \in G$ the equation $ax = b$ has exactly one solution $x \in G$. Do the same for the equation $xa = b$.

(b) Deduce from (b) that every row and column of the multiplication table of G contains exactly one element of G (Sudoku puzzle property).

Problem 6: *This problem describes an important class of matrix groups.* Let F be a field. Denote by $GL_2(F)$ the set of all **invertible** 2×2 matrices with coefficients in F. It is not hard to show that $GL_2(F)$ is a group with respect to matrix multiplication (the the identity element of $GL_2(F)$ is the identity matrix, and the inverse of $A \in GL_2(F)$ is the inverse matrix in the usual sense). In order to determine whether a 2×2 matrix A lies in $GL_2(F)$ one can use the following result from linear algebra:

Theorem. *Let F be a field and let $n \geq 2$ be an integer. Then an $n \times n$ matrix $A \in Mat_n(F)$ is invertible if and only if* $\det(A) \neq 0$.

Also recall that the determinant of a 2×2 matrix is given by the formula

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Thus, $GL_2(F) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in F \text{ and } ad - bc \neq 0. \right\}$

(a) Prove (by direct computation) the following formula for the inverses in $GL_2(F)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Recall that if $\lambda \in F$ is a scalar, then by definition $\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$

(b) Let $F = \mathbb{Z}_7$ and $A = \begin{pmatrix} [1] & [2] \\ [3] & [4] \end{pmatrix}$. Find A^{-1} (and simplify your answer). Answer the same question for $F = \mathbb{Z}_5$.

Problem 7: Section 3.2: 12 (page 151).

Problem 8: Let $G = \{r_0, r_1, r_2, r_3, s_1, s_2, s_3, s_4\}$ be the group of 8 xy-plane transformations preserving a square in $\mathbb{R}^2$. Recall that r_0 is the identity transformation, r_1, r_2, r_3 are counterclockwise rotations by 90, 180 and 270 degrees, respectively, and s_1, s_2, s_3, s_4 are reflections with respect to $y = 0$, $y = x$, $x = 0$ and $y = -x$, respectively.

(a) Prove that G is non-commutative.

(b) Let $H = \{r_0, r_2, s_1, s_3\}$ (so H is a subset of G containing 4 elements). Prove that H is a subgroup of G and compute the multiplication table of H. The definition of a subgroup is given in Section 3.3 and will be given in class on Tuesday, September 28. At this point proving that H is a subgroup will probably have to be done by computing all 16 products directly (but only 4 of those 16 computations will require work).

(c) Show that if we redenote the elements r_0, r_2, s_1, s_3 of H by new symplols a, b, c and d in the suitable order, then the multiplication table for H will coincide with the one for the group from Problem 3.2.11 (we will do this problem in class on Tuesday; the answer is also on the back of the book). This implies that the group from Problem 3.2.11 is isomorphic to H (informally, this means that the two groups have the same properties and differ from each other by relabeling of elements).