**Homework #12. Due Tuesday, December 7th**

**Reading:**

For this assignment: Sections 5.1, 6.1, 6.2.

**Problems:**

**Problem 1:** Let $R$ be a commutative ring with 1 and $I$ an ideal of $R$.

(a) Suppose that $1 \in I$. Prove that $I = R$.

(b) Suppose that $R$ is a field. Prove that $I = R$ or $I = \{0\}$. **Hint:** Reduce (b) to (a).

**Problem 2:** Let $R$ be a commutative ring with 1.

(a) (practice) Fix $a \in R$, and let $I = aR$, the principal ideal of $R$ generated by $a$. Prove that $I$ is the minimal ideal of $R$ containing $a$.

(b) Now fix two elements $a, b \in R$, and let

$$I = aR + bR = \{x \in R : x = ar + bs \text{ for some } r, s \in R\}.$$

Prove that $I$ is the minimal ideal of $R$ containing $a$ and $b$.

**Hint:** First prove that $I$ is an ideal of $R$ containing $a$ and $b$ and then show that if $J$ is any ideal of $R$ containing $a$ and $b$, then $J$ contains $I$.

**Problem 3:** Let $a, b \in \mathbb{Z}$, and let $I$ be the minimal ideal of $\mathbb{Z}$ containing both $a$ and $b$. Use Problem 2 and one of the problems from Homework#2 to prove that $I = d\mathbb{Z}$ where $d = gcd(a, b)$. State your argument clearly.

**Remark:** If $a_1, \ldots, a_k$ are elements of a ring $R$, the minimal ideal of $R$ containing all these elements is commonly denoted by $(a_1, \ldots, a_k)$. With this notation, the result of Problem 3 maybe restated as $(a, b) = (d)$ where $d = gcd(a, b)$. This explains why the greatest common divisor of $a$ and $b$ is often denoted simply by $(a, b)$.

**Problem 4:** Let $R = \mathbb{Z}[x]$ (polynomials with coefficients in $\mathbb{Z}$), and let

$$I = \{a_0 + a_1 x + \ldots + a_n x^n : \text{ each } a_i \in \mathbb{Z} \text{ and } a_0 \text{ is even. } \}$$

(a) Use Problem 2 to prove that $I$ is the minimal ideal of $R$ containing 2 and $x$.

(b) Prove that $I$ is a non-principal ideal, that is, $I \neq fR$ for any $f \in R$.
**Hint:** Consider three cases.

(i) $f$ is a non-constant polynomial

(ii) $f$ is an even constant

(iii) $f$ is an odd constant.

**Problem 5 (practice):** Consider the quotient group $\mathbb{Q}/\mathbb{Z}$. Recall that the group operation on $\mathbb{Q}/\mathbb{Z}$ is denoted by $+$ and defined by

$$(a + \mathbb{Z}) + (b + \mathbb{Z}) = (a + b) + \mathbb{Z}.$$

Suppose now we want to turn $\mathbb{Q}/\mathbb{Z}$ into a ring and define multiplication on $\mathbb{Q}/\mathbb{Z}$ by

$$(a + \mathbb{Z}) \cdot (b + \mathbb{Z}) = ab + \mathbb{Z}.$$

Show that such multiplication will NOT be well defined.

**Bonus Problem:** Give a different proof of the isomorphism

$$\mathbb{R}[x]/(x^2 + 1)\mathbb{R}[x] \cong \mathbb{C}$$

using FTH for rings. See Lecture 26 for a hint on how to start.