Homework #11. Due Thursday, December 2nd

Reading:

1. For this assignment: Sections 4.6 and 5.1 + online notes from Lectures 22 and 23.
2. For the class on Nov 23: Section 5.1 (rings)
3. For the class on Nov 30: Section 6.1 (ideals)
4. For the class on Dec 2: Section 6.1 (quotient rings)

Problems:

Problem 1: Let $G = D_8$, the octic group, and $H = \{r_0, r_2\}$. Describe the elements of the quotient group G/H and compute the multiplication table for G/H. Show details of your computation (recall that we did a sample computations in class). Make sure that in the multiplication table you do not use multiple names for the same element of G/H.

Problem 2: Let $G = \mathbb{Z}_{12}$ and $H = \langle[4]\rangle$, the cyclic subgroup generated by $[4]$.

(a) Describe the elements of the quotient group G/H and compute the "multiplication" table for G/H (the word "multiplication" is in quotes because the group operation in G is addition).

(b) Deduce from your computation in (a) that G/H is isomorphic to $\mathbb{Z}_4$.

(c) Now give a different proof of the isomorphism $G/H \cong \mathbb{Z}_4$ using FTH.

Problem 3: Let A and B be a groups and $G = A \times B$ their direct product. Let $\widetilde{A} = \{(a, e_B) : a \in A\}$ be the subset of G consisting of all elements whose second component is identity. Use FTH to prove that $\widetilde{A}$ is a normal subgroup of G and the quotient group $G/\widetilde{A}$ is isomorphic to B.

Problem 4: This problem deals with the group $\mathbb{Q}/\mathbb{Z}$, the quotient of the group $(\mathbb{Q}, +)$ of rationals with addition by the subgroup of integers.

(a) Prove that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order.

(b) Find all elements of order 12 in $\mathbb{Q}/\mathbb{Z}$ and prove your answer.

Warning: Since elements of quotient groups are defined as cosets, it is common to mispinterpret the notion of the order for such element as the size (cardinality) of the corresponding coset. This is NOT the right interpretation. By the order here we mean the usual notion of the order of group elements (the minimal n such that ...).

Problem 5: Before doing this problem read the full subsection on transversals in the online version of Section 23 (we only discussed part of it in class).

In each of the following examples, find a transversal of H in G. Also decide whether there exists a transversal which is a subgroup: if yes, exhibit such a transversal; if not, prove why.

(a) $G = \mathbb{Z}_6$, $H = \langle [2] \rangle$.

(b) $G = \mathbb{Z}_9$, $H = \langle [3] \rangle$.

(c) $G = D_8$, $H = \{r_0, r_1, r_2, r_3\}$, the rotation subgroup.

(d) $G = D_8$, $H = \{r_0, r_2\}$. **Hint:** Use classification of subgroups of D_8.

Problem 6:

(a) Let $\mathbb{Z}[i]$ be the set of all complex numbers of the form $a + bi$ with $a, b \in \mathbb{Z}$. Prove that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. This ring is called **Gaussian integers.**

(b) (optional) Let $\mathbb{Q}[i]$ be the set of all complex numbers of the form $a+bi$ with $a, b \in \mathbb{Q}$. Prove that $\mathbb{Q}[i]$ is a subfield of $\mathbb{C}$. **Note:** If F is a field and S is a subset of F, to prove that S is a subfield you need to check that S is a subring and, in addition, S contains multiplicative inverses of all its nonzero elements (for any nonzero $s \in S$, the mupltiplicative inverse s^{-1} exists in F because F is a field, but you have to show that s^{-1} actually lies in S).

Problem 7: Let $S = \{a + b\sqrt{2} + c\sqrt{3} : a, b, c \in \mathbb{Z}\}$.

(a) Let T be a subring of $\mathbb{R}$ which contains 1 and $\sqrt{2}$ and $\sqrt{3}$. Prove that T contains all elements of S.

(b) Prove that S is NOT a subring of $\mathbb{R}$.

(c) Find the minimal subring of $\mathbb{R}$ which contains all elements of S. First guess what the answer should be, call your answer S_1 (step 1), then prove that S_1 is a subring (step 2), and finally prove that S_1 is the minimal subring containg S (step 3).

Note: Unlike the example we did in class, your argument in step 1 will likely not be completely rigorous, so some work will have to be done in step 3.

Hint: Your proof in part (b) should suggest which elements must be added to S to get a subring.

Problem 8: Let $R = \mathbb{R}[x]$, the ring of polynomials with real coefficients.

(a) Let $S_0 = \{a_0 + a_2x^2 + a_3x^3 + \ldots + a_nx^n : a_i \in \mathbb{R} \text{ for each } i\}$ be the set of all polynomials with zero coefficient of x. Prove that S_0 is a subring of R.

(b) Let T be any subring of R containing $1, x^2$ and x^3. Prove that $x^k \in T$ for any $k \geq 2$.

(c) Now let S be the minimal subring of R containing 1, x^2 and x^3. Describe S explicitly. Is $S = S_0$?

Bonus Problem: Let V_4 be the Klein four group inside S_4, that is,

$$V_4 = \{e,\ (1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3)\}.$$

Recall that we proved in Lecture 21 that V_4 is a normal subgroup of S_4. Prove that $S_4/V_4 \cong S_3$ without using classification of groups of order 6. **Hint:** FTH is probably not the best way to go here; however, there is another result from Lecture 23 that will help a lot.