Homework #10. Due Thursday, November 18th

Reading:

1. For this assignment: Sections 4.4, 4.5 and the second part of 4.1 (even/odd permutations and conjugacy classes in S_n) + online notes
2. For the classes on Nov 16 and 18: Section 4.6 (quotient groups)

Problems:

Problem 1: Let G be a group and H a subgroup of G. Consider the following relation $\sim$ on G:

$$g \sim k \iff g^{-1}k \in H.$$

(i) Prove that $\sim$ is an equivalence relation.

(ii) Prove that for every $g \in G$ its equivalence class with respect to $\sim$ coincides with gH, the left coset of g with respect to H.

Problem 2: Let G be a group and H a subgroup of G. In each of the following examples describe left cosets of G with respect to H. Find the number of distinct cosets and list all elements in each coset.

(a) $G = \mathbb{Z}_{12}$, $H = \langle [3] \rangle$.

(b) $G = D_8$ (the octic group), $H = \{r_0, r_1, r_2, r_3\}$ (the rotation subgroup).

(c) $G = D_8$, $H = \langle s_1 \rangle = \{r_0, s_1\}$ (recall that s_1 is the reflection wrt $y = 0$).

Problem 3: Let G be a group and H a subgroup of G.

(a) Let $g \in G$. Prove that $gH = H$ if and only if $g \in H$. (**Hint:** This is not hard to prove directly, but the result follows easily from Proposition 19.2 or from Problem 1(b)). State the analogous result for right cosets.

(b) Suppose that H has index 2 in G. Prove that H is normal in G (you will likely need (a) for your proof). **Note:** Usually, to prove that a subgroup is normal, conjugation criterion (Theorem 20.2) is easier to use than definition, but this problem is a rare exception. **Hint:** see the end of the assignment.

Problem 4: Let $G = D_8$. For each subgroup of D_8, determine whether it is normal or not (for the complete list of subgroups of D_8 see solutions to homework#9). **Hint:** For subgroups which are normal, use one of the criteria discussed in class. Note that the center $Z(G) = \{r_0, r_2\}$ (verify this). For subgroups which are not normal, give a direct proof that they are not normal (using definition).

Problem 5: Let F be a field. Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in F \text{ and } ad \neq 0 \right\}$, and let $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F \right\}$.

(a) Prove that B is a subgroup of $GL_2(F)$. Recall that U was shown to be a subgroup of $GL_2(F)$ in Homework#6. Since U is clearly a subset of B, we conclude that U is also a subgroup of B.

(b) Use the conjugation criterion (Theorem 20.2) to prove that U is a normal subgroup of B.

(c) Prove that U is NOT a normal subgroup of $GL_2(F)$.

Note: You will probably need the formula for inverses in $GL_2(F)$ given in Problem 6(a) of Homework#5.

Problem 6: Before doing this problem read about even and odd permutations either in the book (pp. 196-199) or in the online notes.

(a) Write the permutation $(1,2)(3,4,5)(6,7,8,9)(10,11,12)(13,14)$ as a product of transpositions.

(b) Let $f \in S_n$ be a cycle of length k. Prove that f is even if k is odd, and f is odd if k is even.

(c) Let $f \in S_n$. Write f as a product of disjoint cycles $f = f_1 f_2 \ldots f_r$, and let k_i be the length of f_i for each i. Suppose that the "length sequence" $\{k_1, k_2, \ldots, k_r\}$ contains a even numbers and b odd numbers. For instance, the length sequence of the permutation in part (a) is $\{2, 3, 4, 3, 2\}$, so $a = 3$ and $b = 2$.

Among the following 4 statements exactly one is correct. Find the correct statement and prove it.

(i) f is even if and only if a is even

(ii) f is even if and only if a is odd

(iii) f is even if and only if b is even

(iv) f is even if and only if b is odd

Problem 7:

(a) Consider the permuations $g = (1, 3, 5)(2, 4, 7, 8)$ and $f = (1, 7, 5, 6)(2, 8, 9)(3, 4)$ in S_9. Compute gfg^{-1} (you should be able to write down the answer right away).

(b) Consider the permutations $f = (1, 4, 6)(2, 3, 5)$ and $h = (3, 4, 6)(1, 5, 7)$ in S_7. Find $g \in S_7$ such that $gfg^{-1} = h$, $g(1) = 1$ and $g(3) = 3$.

(c) Let $f = (1, 2, 3)$ considered as an element of S_6, and let $C(f)$ be the centralizer of f in S_6. Prove that $|C(f)| = 18$. **Hint:** Use the conuga-tion formula.

Problem 8:

(a) (practice) List explicitly all elements of the alternating group A_4, that is, all even permutations in S_4. **Note:** The answer is contained in the main text of the book, so do this problem without using the book.

(b) Prove that S_5 has 7 conjugacy classes, and the sizes of the conjugacy classes are $1, 10, 15, 20, 20, 24$ and 30.

(c) Use Theorem 21.3 to prove that the only normal subgroups of S_5 are S_5, $\{e\}$ and A_5.

Bonus Problem: The goal of this problem is provide of a different proof of the fact that the notion of even/odd permutation is well defined. Let $n \geq 2$ be an integer.

(a) For each $\sigma \in S_n$ let $P(\sigma) \in GL_n(\mathbb{Z})$ be the $n \times n$ matrix whose (i, j)-entry $P(\sigma)_{ij}$ is given by

$$P(\sigma)_{ij} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{if } i \neq \sigma(j). \end{cases}$$

Prove that the map $P : S_n \to GL_n(\mathbb{Z})$ given by $\sigma \mapsto P(\sigma)$ is a homo-morphism.

(b) Suppose that $\sigma \in S_n$ is a transposition. Prove that $\det P(\sigma) = -1$. **Hint:** The matrix $P(\sigma)$ is obtained from the identity matrix using a simple row operation.

(c) Deduce from (b) that if $\sigma \in S_n$ and σ is written as a product of trans-positions in two different ways: $\sigma = \tau_1 \ldots \tau_k$ and $\sigma = \tau'_1 \ldots \tau'_l$, then k and l are both even or both odd.

Hint for Problem 4: Since H has index 2 in G, there are only two left cosets, one of which is H itself – use this to describe the other coset. Then do the same for right cosets. Now recall that we need to prove $xH = Hx$ for every $x \in G$. Consider two cases: $x \in H$ and $x \notin H$.