Homework #1. Due TUESDAY, August 31st, in class

Reading:

1. For this homework assignment: Sections 2.1, 2.2 and 1.2. A good supplement to Section 1.2 is provided by *Wikipedia* - look at the articles "Injective function", "Surjective function" and "Bijective function."

2. Before the class on Tue, Aug 31st: Section 2.3. Before the class on Thu, Sep 2nd: Section 2.4.

Problems:

Problem 1: Let R be a commutative ring with 1. Prove the following equalities using only the ring axioms and results proved or stated in class.

(a) $-(-x) = x$ for all $x \in R$ (GG 2.1.3)

(b) $(-1)(-1) = 1$ (GG 2.1.4)

(c) $(-x)(-y) = xy$ for all $x, y \in R$ (GG 2.1.5)

(d) $x(y - z) = xy - xz$ for all $x, y, z \in R$ (GG 2.1.7)

Hint: Use the cancellation law (Lemma 2.3) and the identity $(-x) = (-1) \cdot x$ (to be proved in class). Recall that by definition $x - y = x + (-y)$.

Problem 2: Let F be a field, and suppose that $xy = 0$ for some $x, y \in F$. Prove that $x = 0$ or $y = 0$. **Hint:** Consider two cases: $x = 0$ (in this case there is nothing to prove) and $x \neq 0$. Recall that in a field every nonzero element has multiplicative inverse.

Note: If F was only assumed to be a commutative ring with unity, the above assertion would have been false in general. Can you think of an example?

Problem 3: For each of the following functions determine whether it is injective and whether it is surjective. Prove your answer.

(a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$.

(b) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$.

(c) $f : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ given by $f(x) = x^2$ where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers.

(d) $f : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^2$.

Hint: The answers in (a) and (b) are different, as well as the answers in (c) and (d).

Problem 4: Let R be an ordered ring and $x, y, z \in R$. Prove that

(a) If $x < y$, then $x + z < y + z$ (GG 2.1.13)

(b) If $x < y$ and $z < 0$, then $xz > yz$ (GG 2.1.19)

Note: You may use freely standard properties of ring operations (addition, subtraction and multiplication). However, all statement involving inequalities must be deduced directly from the axioms.

Problem 5: Prove by induction that the following equalities hold for any $n \in \mathbb{Z}_{>0}$:

(a) $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ (GG 2.2.3)

(b) $a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{1-r^n}{1-r}$ where $a, r \in \mathbb{R}$ and $r \neq 1$ (GG 2.2.14)

Problem 6: Prove the following inequalities by generalized induction (in both (a) and (b) n is assumed to be an integer):

(a) $1 + 2n < 2^n$ for any $n \geq 3$ (GG 2.2.39)

(b) $n^2 < 2^n$ for any $n \geq 5$ (GG 2.2.43)

Problem 7: Consider the following "proof" by induction: For each $n \in \mathbb{Z}_{>0}$ let P_n be the statement

$$\sum_{i=0}^{n} 2^i = 2^{n+1}. \qquad (***)$$

Claim: P_n is true for all $n \in \mathbb{Z}_{>0}$.

Proof: "$P_{n-1} \Rightarrow P_n$." Assume that P_{n-1} is true for some $n \in \mathbb{Z}^+$. Then $\sum_{i=0}^{n-1} 2^i = 2^n$. Adding 2^n to both sides, we get $\sum_{i=0}^{n-1} 2^i + 2^n = 2^n + 2^n$, whence $\sum_{i=0}^{n} 2^i = 2^{n+1}$, which is precisely P_n. Thus, P_n is true.

By the principle of mathematical induction, P_n is true for all n. □

(a) Show that the statement P_n is false (it is actually false for any n).

(b) Explain why the above "proof" does not contradict the principle of mathematical induction, that is, find a mistake in the above "proof" (Hint: the mistake is in the general logic).

Problem 8: Prove by induction that for any $n \in \mathbb{Z}_{>0}$ there exist integers a_n and b_n such that

$$(1 + \sqrt{2})^n = a_n + b_n\sqrt{2}.$$

Hint: For the induction step write $(1+\sqrt{2})^n$ as $(1+\sqrt{2})^{n-1} \cdot (1+\sqrt{2})$. Note that you do not have to find a formula for a_n and b_n; you just have to show they exist. It may be helpful to start by computing $(1+\sqrt{2})^n$ directly for small n (say, $n = 1, 2, 3, 4$) to understand intuitively why the above statement is true.