13. A note on writing negations

Statements in mathematical analysis typically involve lots of quantifiers, that is, symbols $\forall$ (for all) and $\exists$ (there exists). If you are asked to formulate a negation of a statement of this kind and you do not feel completely comfortable with negations, it may be a good idea to proceed one step at a time using the following basic principles:

(a) Suppose the statement P we want to negate has the form

$$P = (\forall t \in A \quad Q(t))$$

where A is some set and $Q(t)$ is a substatement depending on t. The statement P asserts that $Q(t)$ should be true for ALL $t \in A$. Hence its negation $\neg P$ should say that there is AT LEAST ONE $t \in A$ such that $Q(t)$ is false. In other words,

$$\neg P = \neg(\forall t \in A \quad Q(x)) = (\exists t \in A \quad \neg Q(t))$$

(b) Suppose now the statement P has the form

$$P = (\exists t \in A \quad Q(t)).$$

Now P asserts that $Q(t)$ should be true for AT LEAST ONE $t \in A$. Hence its negation $\neg P$ should say that $Q(t)$ is false for ALL $t \in A$. In other words,

$$\neg P = \neg(\exists t \in A \quad Q(t)) = (\forall t \in A \quad \neg Q(t))$$

Let us now use these principles to see how to negate the statement $\lim\limits_{x \to a} f(x) = L$ (where $a, L \in \mathbb{R}$ and f is some real function defined near $a \in \mathbb{R}$); recall that this negation was needed for the proof of reverse direction of Theorem 13.4 from class.

So, the original statement P is $P = (\lim\limits_{x \to a} f(x) = L)$ which using the definition of limit becomes

$$P = (\forall \varepsilon > 0\, \exists \delta > 0 \text{ s.t. } |f(x) - L| < a \text{ for all } x \text{ s.t. } 0 < |x - a| < \delta).$$

To apply the above negation principles in this case, we need to slightly rephrase P, for which it is convenient to introduce the following notations: let $\mathbb{R}_{>0}$ denote the set of all positive real numbers, and given $a, \delta \in \mathbb{R}$, let $B^{\circ}_{\delta}(a) = \{x \in \mathbb{R} : 0 < |x - a| < \delta\} = (a - \delta, a + \delta) \setminus \{a\}$. Then we can rephrase P as follows:

$$P = (\forall \varepsilon \in \mathbb{R}_{>0}\, \exists \delta \in \mathbb{R}_{>0} \text{ s.t. } \forall x \in B^{\circ}_{\delta}(a) \text{ we have } |f(x) - L| < \varepsilon).$$

Note that $P = (\forall \varepsilon \in \mathbb{R}_{>0} \quad Q(\varepsilon))$ where

$$Q(\varepsilon) = (\exists \delta \in \mathbb{R}_{>0} \text{ s.t. } \forall x \in B^\circ_\delta(a) \text{ we have } |f(x) - L| < \varepsilon).$$

Hence by principle (a) we have $\neg P = (\exists \varepsilon \in \mathbb{R}_{>0} \quad \neg Q(\varepsilon))$. Similarly, we can negate $Q(\varepsilon)$ using principle (b). We get $\neg Q(\varepsilon) = (\forall \delta \in \mathbb{R}_{>0} \quad \neg R(x))$ where

$$R(x) = (\forall x \in B^\circ_\delta(a) \text{ we have } |f(x) - L| < \varepsilon).$$

Finally, again by principle (a) we have

$$\neg R(x) = (\exists x \in B^\circ_\delta(a) \text{ s.t. } \neg(|f(x) - L| < \varepsilon)).$$

The statement $|f(x) - L| < \varepsilon$ does not involve any quantifiers, and we can write its negation directly: $\neg(|f(x) - L| < \varepsilon) = (|f(x) - L| \geq \varepsilon)$.

Putting everything together, we can now write down the negation of the original statement P:

$$\neg P = (\exists \varepsilon \in \mathbb{R}_{>0} \text{ s.t. } \forall \delta \in \mathbb{R}_{>0} \, \exists x \in B^\circ_\delta(a) \text{ s.t. } |f(x) - L| \geq \varepsilon).$$

Now that the negation has been formulated we can get rid of all the extra notations and rephrase $\neg(P)$ as follows:

$$\neg P = (\exists \varepsilon > 0 \text{ s.t. } \forall \delta > 0 \, \exists x \text{ s.t. } 0 < |x - a| < \delta \text{ and } |f(x) - L| \geq \varepsilon).$$

Note that the appearance of the expression "there exists x ..." in certain statement P does not imply that the negation $\neg(P)$ will involve the expression "for all x". Consider the following example.

Example 1. *Suppose A is a subset of $\mathbb{Z}$ (integers) and P is the following statement:*

$$\forall x \in A \textit{ there exist at most 3 primes } p \textit{ s.t. } p \textit{ divides } x.$$

By principle (a), we have $\neg P = (\exists x \in A \text{ s.t. } \neg Q(x))$ where $Q(x)$ is the statement "there exist at most 3 primes p s.t. p divides x". However, we cannot use principle (b) to form negation of $Q(x)$ since $Q(x)$ does not say that there exists a prime p with certain property; in fact, it tells us almost the opposite: there are at most 3 (possibly 0) primes p with certain property. **Hint:** if you do not see how to formulate the negation of $Q(x)$, try to rephrase $Q(x)$ without using the expression "there exists".