Math 3310, Section 1. Spring 2015. First Midterm.
Thursday, April 2nd, 2-3:20 pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts. Write your answers and all accompanying work neatly on these pages.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

1. (9 pts) Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ a continuous function. Prove that f is bounded on $[a, b]$. Clearly indicate where (and how) you use the fact that the domain of f is a closed bounded interval.

2. (a) (3 pts) Give the definition of uniform continuity: Let f be a real function and E a subset of domain of f. Then f is called uniformly continuous on E if ... (complete the sentence).

(b) (5 pts) Prove **directly from definition** that the function $f(x) = \frac{1}{x}$ is uniformly continuous on $[5, 20]$. (Do not refer to any theorems about uniform continuity).

3. (a) (2 pts) State the Intermediate Value Theorem (either basic or general version).

(b) (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x+1)$. Suppose that f is continuous at a for some $a \in \mathbb{R}$. Prove that g is continuous at $a - 1$ directly using ε-δ definition of continuity.

(c) (5 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that $f(0) = f(2)$. Prove that there exists $c \in \mathbb{R}$ such that $f(c) = f(c+1)$.

4. (a) (1 pt) Give the definition of a countable set.

(b) (6 pts) Let A be an uncountable set, and let B and C be countable subsets of A such that $B \cap C = \emptyset$.

(i) (3 pts) Prove that the set $A \setminus (B \cup C)$ is uncountable.

(ii) (3 pts) Prove that there exists a bijection $\phi : A \setminus B \to A$.

Hint: For both parts: draw a picture. For (ii): start with the identity function $id : A \setminus (B \cup C) \to A \setminus (B \cup C)$ and extend it to a bijection from $A \setminus B$ to A.

5. Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function, and suppose that $\lim\limits_{x \to +\infty} f(x)$ exists.

(a) (5 pts) Prove that f is bounded on $[0, +\infty)$.

(b) (5 pts) Let $L = \lim\limits_{x \to +\infty} f(x)$, and assume that there exists a real number $a \geq 0$ s.t. $f(a) > L$. Prove that f attains maximum value, that is, there exists $c \in [0, +\infty)$ s.t. $f(c) \geq f(x)$ for all $x \in [0, +\infty)$.