Math 3310, Section 2. Spring 2015. First Midterm.
Thursday, February 19th, 2-3:20 pm

Directions: No books, notes, calculators, laptops, PDAs, cellphones, web appliances, or similar aids are allowed. All work must be your individual efforts. Write your answers and all accompanying work neatly on these pages.

- Show all your work and justify all statements that you make.
- State clearly and fully any theorem you use.
- Vague statements and hand-waving arguments will not be appreciated.
- You may assume the statement in an earlier part proven in order to do a later part.

1. (12 pts) Let $I_1 \supseteq I_2 \supseteq \ldots$ be a descending chain of non-empty closed bounded intervals in $\mathbb{R}$. Prove that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. Moreover prove that if $length(I_n) \to 0$ as $n \to \infty$, then $\cap_{n=1}^{\infty} I_n \neq \emptyset$ is a single point.

Note: You may use the comparison and monotone convergence theorems. CLEARLY justify every step.

2. (6 pts) Prove by induction that $4^n > 2^n + 3^n$ for all $n \geq 2$. You do not need to explicitly refer to axioms, but all the steps should be clear.

3. (8 pts) Let $A_1, A_2, \ldots$ be an (infinite) sequence of subsets of $\mathbb{R}$ and let $A = \cup_{n=1}^{\infty} A_n$. Assume that A is bounded above (so that each A_n is bounded above as well). For each $n \in \mathbb{N}$ let $M_n = \sup(A_n)$ and let

$$S = \{M_1, M_2, \ldots\}.$$

Prove that

$$S \text{ is also bounded above and } \sup(S) = \sup(A).$$

Provide ALL details. **Note:** Partial credit will be given for proving either of the inequalities $\sup(S) \leq \sup(A)$ or $\sup(A) \leq \sup(S)$.

4. (8 pts) Let $\{a_n\}$ be a sequence, and suppose that for every $\varepsilon > 0$ we have $|a_n - 4| < \varepsilon$ for all $n > 7 + \frac{10}{\varepsilon^2}$.

(a) (3 pts) Find explicit numbers $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|a_n| \leq C$ for all $n \geq N$

(b) (5 pts) For every $\varepsilon > 0$ find an explicit real number $M(\varepsilon)$ such that $|a_n^2 - 16| < \varepsilon$ for all $n > M(\varepsilon)$.

In both parts justify your claim and include all the computations.

5. In both parts of this problem $\{a_n\}$ is a sequence and L is a real number, but assumptions about $\{a_n\}$ in L in (a) and (b) are independent of each other.

(a) (3 pts) Suppose that for every $\varepsilon > 0$ there are only finitely many $n \in \mathbb{N}$ such that $|a_n - L| \geq \varepsilon$. Prove (directly from the definition of limit) that $\{a_n\}$ converges to L.

(b) (7 pts) Suppose that $\{a_n\}$ is bounded. Prove that precisely one of the following holds:

(i) $\{a_n\}$ converges to L

(ii) $\{a_n\}$ has a subsequence which converges to some real number different from L

In other words, prove that (i) and (ii) cannot hold simultaneously (2 pts) and that (i) or (ii) must hold (5 pts). **Hint:** to prove the latter, assume that (i) does not hold and use (a) (in the contrapositive form) to prove that (ii) holds.