15. Intermediate Value Theorem and Classification of discontinuities

15.1. **Intermediate Value Theorem.** Let us begin by recalling the definition of a function continuous at a point of its domain.

Definition. Let $f : E \to \mathbb{R}$ be a real function and $a \in E$. We say that f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|f(x) - f(a)| < \varepsilon \text{ for all } x \in E \text{ s.t. } |x - a| < \delta;$$

$$\text{equivalently, } |f(x) - f(a)| < \varepsilon \text{ for all } x \in (a - \delta, a + \delta) \cap E. \qquad (***)$$

In this lecture we will usually apply the definition of continuity in the form (***).

Our first result asserts that a function must preserve sign near any point where it is continuous. In other words, if f is continuous at a and $f(a)$ is positive (respectively, negative), then $f(x)$ must be positive (respectively, negative) for all x in the domain of f which are sufficiently close to a:

Lemma 15.1 (Sign preservation Lemma)**.** *Suppose that $f : E \to \mathbb{R}$ is continuous at some $a \in \mathbb{R}$.*

(a) *If $f(a) > 0$, then there exists $\delta > 0$ s.t. $f(x) > 0$ for all $x \in (a - \delta, a + \delta) \cap E$;*
(b) *If $f(a) < 0$, then there exists $\delta > 0$ s.t. $f(x) < 0$ for all $x \in (a - \delta, a + \delta) \cap E$.*

Proof. We will prove (a); the proof of (b) is completely analogous. Applying the definition of continuity with $\varepsilon = f(a)$, we get that there exists $\delta > 0$ s.t. $|f(x) - f(a)| < f(a)$ for all $x \in (a - \delta, a + \delta) \cap E$. Since $|f(x) - f(a)| < f(a)$ is equivalent to $-f(a) < f(x) - f(a) < f(a) \iff 0 < f(x) < 2f(a)$, the result follows. $\square$

We are now ready to state and prove the intermediate value theorem. Recall that a function is called continuous (on its domain) if it is continuous at every point of its domain.

Basic Intermediate Value Theorem. *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that ($f(a) < 0$ and $f(b) > 0$) or ($f(a) > 0$ and $f(b) < 0$). Then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We will prove the theorem in the case $f(a) < 0$ and $f(b) > 0$. The proof in the other case is analogous. Let

$$S = \{x \in [a, b] : f(x) < 0\}.$$

By assumption $a \in S$ (so S is non-empty), and by definition S is bounded above (by b). Hence $\sup(S)$ exists (and moreover $a \leq \sup(S) \leq b$). We will prove that $f(\sup(S)) = 0$.

First, we shall show that $a < \sup(S) < b$. Applying the Sign Preservation Lemma at a (the left endpoint of the interval) we conclude that there exists $\delta_1 > 0$ s.t. $f(x) < 0$ for all $x \in (a - \delta_1, a + \delta_1) \cap [a, b]$ (here $E = [a, b]$). We can always make δ_1 smaller, so in particular we can require that $\delta_1 < b - a$. This ensure that $(a - \delta_1, a + \delta_1) \cap [a, b] = [a, a + \delta_1)$, and therefore $f(x) < 0$ for all $x \in [a, a + \delta_1)$. This means that the half-closed interval $[a, a + \delta_1)$ is contained in S, whence $\sup(S) \geq \sup[a, a + \delta_1) = a + \delta_1 > a$.

Similarly, applying the Sign Preservation lemma at b (the right endpoint of the interval) we conclude that there exists $\delta_2 > 0$ s.t. $f(x) > 0$ for all $x \in (b - \delta_2, b]$. This means that the intersection $S \cap (b - \delta_2, b]$ is empty, so $S \subseteq [a, b - \delta_2]$ and therefore $\sup(S) \leq b - \delta_2 < b$.

Thus, we showed that $a < \sup(S) < b$. We shall now prove that $f(\sup(S)) = 0$ by contradiction. Assume that $f(\sup(S)) \neq 0$, so either $f(\sup(S)) < 0$ or $f(\sup(S)) > 0$. In each case we will obtain a contradiction.

Case 1: $f(\sup(S)) < 0$. Applying the Sign Preservation Lemma at $\sup(S)$, we conclude that there exists $\delta_3 > 0$ s.t. $f(x) < 0$ for all $x \in (\sup(S) - \delta_3, \sup(S) + \delta_3) \cap [a, b]$. Since $\sup(S)$ is not an endpoint of $[a, b]$, after making δ_3 smaller we can assume that $(\sup(S) - \delta_3, \sup(S) + \delta_3) \subseteq [a, b]$ (it is enough to require that $\delta_3 < \min\{\sup(S) - a, b - \sup(S)\}$). Then $f(x) < 0$ for all $x \in (\sup(S) - \delta_3, \sup(S) + \delta_3)$; in particular, $f(\sup(S) + \frac{\delta_3}{2}) < 0$, so $\sup(S) + \frac{\delta_3}{2} \in S$, which means that $\sup(S)$ is not an upper bound for S, a contradiction.

Case 2: $f(\sup(S)) > 0$. Similarly, by the Sign Preservation Lemma there exists $\delta_4 > 0$ s.t. $(\sup(S) - \delta_4, \sup(S) + \delta_4) \subseteq [a, b]$ and $f(x) > 0$ for all $x \in (\sup(S) - \delta_4, \sup(S) + \delta_4)$. This means that there is NO $x \in S$ s.t. $\sup(S) - \delta_4 < x \leq \sup(S)$, which contradicts the approximation theorem for suprema. $\square$

Remark: Note that in the above proof we had to consider both cases 1 and 2; we could not say WOLOG $f(\sup(S)) < 0$. The reason for the latter is that while the initial hypotheses in the theorem were symmetric (with respect to swapping inequalities < 0 and > 0), we broke the symmetry when defining the set S (this breaking of symmetry was necessary). In fact, as you can see, the contradictions we got in cases 1 and 2 were not identical, so the argument in case 2 was not a mere repetition of the argument in case 1.

We now state and prove the general version of the intermediate value theorem, which easily follows from the special case proved above.

Intermediate Value Theorem (IVT). *Let $a < b$ be real numbers, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. If y is any real number which lies between $f(a)$ and $f(b)$ (that is, $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$), then there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof. If $y = f(a)$ or $y = f(b)$, there is nothing to prove, so we can assume that $f(a) < y < f(b)$ or $f(b) < y < f(a)$. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - y$. Then g is continuous (by Theorem 14.3) and by construction either ($g(a) < 0$ and $g(b) > 0$) or ($g(a) > 0$ and $g(b) < 0$). Thus, we can apply basic IVT to the function g to conclude that there is $c \in (a, b)$ s.t. $g(c) = 0$ which means $f(c) = y$. $\square$

15.2. **Classification of discontinuities.** For the discussion below we assume that $a \in \mathbb{R}$ is fixed and f is a real function defined near a (that is, there exists $\delta > 0$ s.t. $(a - \delta, a + \delta) \subseteq \text{domain}(f)$). Recall that in this situation f is continuous at $a \iff \lim\limits_{x \to a} f(x)$ and $f(a)$ are both defined and $\lim\limits_{x \to a} f(x) = f(a)$. Thus, f can be discontinuous at a for two reasons:

(1) $\lim\limits_{x \to a} f(x)$ exists (as a finite limit), but either $\lim\limits_{x \to a} f(x)$ does not equal $f(a)$ or $f(a)$ is not defined

(2) $\lim\limits_{x \to a} f(x)$ does not exist.

Case (2) can be naturally separated into two subcases. As proved in § 3.2 of the book, $\lim\limits_{x \to a} f(x)$ exists $\iff$ both one-sided limits $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ exist and equal each other. Thus, $\lim\limits_{x \to a} f(x)$ may not exist for two reasons:

(2a) $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist (as finite limits), but are not equal to each other.

(2b) at least one of the one-sided limits $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ does not exist.

This analysis leads to the following division of all discontinuities into three types.

Definition. In the above setting we say that

(a) f has a type 0 (removable) discontinuity at a if $\lim_{x\to a} f(x)$ exists (as a finite limit), but either $\lim_{x\to a} f(x)$ does not equal $f(a)$ or $f(a)$ is not defined.

(b) f has a type 1 (jump) discontinuity at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist (as finite limits), but are not equal to each other

(c) f has a type 2 discontinuity at a if at least one of the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ does not exist.

Example 1.
(a) *$f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ given by $f(x) = \frac{x^2-1}{x-1}$ has a removable discontinuity at 1 since $f(x) = x + 1$ for all $x \neq 1$ (so $\lim_{x\to 1} f(x)$ exists);*
(b) *$f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ for $x \geq 0$ and $f(x) = -1$ for $x < 0$ has a jump discontinuity at 0;*
(c) *$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ and $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by $g(x) = sin(\frac{1}{x})$ both have type 2 discontinuity at 0.*

15.3. **Some strongly discontinuous functions.** The functions studied in calculus are typically continuous at "most" points where they are defined. The following two examples show that functions can have lots of discontinuities.

Example 2. *The Dirichlet function $D : \mathbb{R} \to \mathbb{R}$ is defined by*

$$D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Theorem 15.2. *The Dirichlet function D is discontinuous at every $a \in \mathbb{R}$.*

The proof of Theorem 15.2 is based on the following lemma:

Lemma 15.3. *Let $a \in \mathbb{R}$. Then there exists a sequence $\{x_n\}$ of rational numbers which converges to a and a sequence $\{y_n\}$ of irrational numbers which converges to a.*

Proof. The existence of a sequence $\{x_n\}$ of rational numbers s.t. $x_n \to a$ was proved in HW#5.4. The existence of a sequence $\{y_n\}$ of irrational numbers s.t. $y_n \to a$ is proved similarly. Indeed, the only fact about rational numbers used in the solution to HW#5.4 was their density in $\mathbb{R}$, namely, the fact that any closed bounded interval $[c, d]$, with $c < d$, contains a rational number. But we also know that $[c, d]$ must contain an irrational number (if not, all real numbers in $[c, d]$ would be rational, hence $[c, d]$ would be countable, contrary to what we proved in Lecture 12). Thus, the argument from the solution to HW#5.4 also shows that there is a sequence $\{y_n\}$ of irrational numbers which converges to a. $\square$

Proof of Theorem 15.2. We argue by contradiction. Suppose that D is continuous at some $a \in \mathbb{R}$. Then by sequential characterization of continuity (Theorem 14.2), for any sequence $\{a_n\}$ which converges to a, we must have $D(a) = \lim_{n\to\infty} D(a_n)$.

If $a \in \mathbb{Q}$, choose a sequence $\{y_n\}$ of irrational numbers such that $y_n \to a$ (which exists by Lemma 15.3). Then $D(y_n) = 0$ for all n, so $\lim_{n\to\infty} D(y_n) = 0 \neq 1 = D(a)$, a contradiction. Similarly, if $a \notin \mathbb{Q}$, we choose a sequence $\{x_n\}$ of rational numbers such that $x_n \to a$ to obtain a contradiction. $\square$

Example 3. *The modified Dirichlet function $MD : \mathbb{R} \to \mathbb{R}$ is defined by*

$$MD(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms (by convention } q > 0) \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Theorem 15.4. *The modified Dirichlet function MD is discontinuous at every $a \in \mathbb{Q}$ and continuous at every $a \notin \mathbb{Q}$.*

Proof. The fact that MD is discontinuous at every $a \in \mathbb{Q}$ is proved in the same way as Theorem 15.2.

Suppose now that $a \notin \mathbb{Q}$. We shall prove that MD is continuous at a directly using ε-δ definition of limit (see Example 3.33 in the book for a proof using sequential characterization of continuity). We start with a lemma.

Lemma 15.5. *Let $n \in \mathbb{N}$. There exists $\delta > 0$ (depending on n) such that the interval $(a - \delta, a + \delta)$ has NO rational numbers with denominator $\leq n$.*

Proof of the Lemma. Let $I = [a - 1, a + 1]$. Clearly, for every $q \in \mathbb{N}$, the interval I contains only finitely many rational numbers with denominator q. Since there are only finitely many natural numbers $\leq n$, it follows that I contains only finitely rational numbers with denominator $\leq n$. Denote those rational numbers by $c_1, \ldots, c_k$. If we define

$$\delta = \min\{1, |c_1 - a|, \ldots, |c_k - a|\},$$

then the interval $(a - \delta, a + \delta)$ contains NO rational numbers with denominator $\leq n$. Note that $|c_i - a| > 0$ for each i (since a is irrational), which ensures that $\delta > 0$. $\square$

We now use the lemma to finish the proof of Theorem 15.4. Fix $\varepsilon > 0$. Since $MD(a) = 0$ (as $a \notin \mathbb{Q}$), to prove that MD is continuous at a we need to find δ s.t.

$$|MD(x)| < \varepsilon \text{ for all } x \in (a - \delta, a + \delta) \qquad (***)$$

Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. By the Lemma there exists $\delta > 0$ such that $(a - \delta, a + \delta)$ has NO rational numbers with denominator $\leq n$. We will show that this δ satisfies (***).

Take any $x \in (a - \delta, a + \delta)$. If $x \notin \mathbb{Q}$, then $MD(x) = 0$, and there is nothing to prove. So suppose that $x \in \mathbb{Q}$ and write $x = \frac{p}{q}$ in lowest terms. Then $q > n$ by the choice of δ, so $|MD(x)| = |\frac{1}{q}| = \frac{1}{q} < \frac{1}{n} < \varepsilon$, so (***) indeed holds. $\square$