Homework #9. Due Thursday, April 9th, in class

Reading:

1. For this assignment: 4.1-4.3 + class notes (Lectures 17-18).
2. For next week's classes: first part of 4.4 (Taylor's theorem and applications) and 5.1 (the Riemann integral).

Problems:

General hint: Mean Value Theorem (MVT) is applicable to several problems in this assignment.

Problem 1: Prove the power rule: if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$ for some fixed $n \in \mathbb{N}$, then $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. **Hint:** this can be done either by induction or directly from the definition of derivative using binomial formula.

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

(a) Suppose that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that f' is constant.

(b) Suppose that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists a constant C s.t. $f'(x) = Cf(x)$ for all $x \in \mathbb{R}$.

Hint: Use the definition of derivative which involves h approaching zero.

Problem 3: Let I be an interval, let I_0 be I with endpoints removed, and let $f : I \to \mathbb{R}$ be a function. Suppose that f is continuous on I, differentiable on I_0 and there exists $C \in \mathbb{R}$ s.t. $|f'(x)| \leq C$ for all $x \in I_0$. Prove that f is uniformly continuous on I.

Problem 4: A *root* of a real function g is a point x of the domain of g s.t. $g(x) = 0$.

(a) Let I be an interval, let I_0 be I with endpoints removed, and let $f : I \to \mathbb{R}$ be a function which is continuous on I and differentiable on I_0. Suppose that f has at least n roots for some $n \in \mathbb{N}$. Prove that f' has at least $n-1$ roots.

(b) Use (a) and induction to prove that a polynomial of degree n (with real coefficients) has at most n roots.

Hint: for (a) – start with the cases $n = 2$ and $n = 3$ (in the case $n = 1$ there is nothing to prove); drawing a picture will likely be helpful, but of course your formal argument should not reference a picture. For (b): prove the induction step by contradiction.

Problem 5: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and suppose that both limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} f'(x)$ exist (and are finite). Prove that $\lim_{x\to+\infty} f'(x) = 0$.

Hint: Use the assumption that $\lim_{x\to+\infty} f(x)$ exists and MVT to construct a sequence $\{x_n\}$ s.t. $x_n \to +\infty$ and $f'(x_n) \to 0$ (the interval on which MVT is applied would naturally need to depend on n). Then finish the proof using sequential characterization of limits at infinity (special case of Theorem 3.17 in the book).

Problem 6: Prove that $\sqrt{x} + \frac{1}{2}\log(x) \le x$ for all $x \ge 1$. **Note:** here $\log(x)$ is the logarithm to the base e (typically denoted by $\ln(x)$ in calculus). You can assume without proof all the standard properties of log; in particular, the fact that $\log : (0, +\infty) \to \mathbb{R}$ is differentiable and $(\log x)' = \frac{1}{x}$. An example similar to this problem is done in 4.3.