Homework #8. Due Thursday, March 26th, in class

Reading:

1. For this assignment: 3.3, 3.4 + class notes (Lectures 15-16).

2. For next week's classes: 4.1 (definition of derivative), 4.2 (differentiability rules) and 4.3 (the mean value theorem). I am not sure how fast we will proceed, but I hope that we will at least start talking about the mean value theorem.

Problems:

Problem 1: In each part of this problem prove that the given function f is uniformly continuous on the given set E directly from definition and give an explicit formula for δ in terms of ε.

(a) $f(x) = x^3$, $E = [1, 2]$
(b) $f(x) = \sqrt{x}$, $E = [0, \infty)$.

Hint: In (b), once you fixed $\varepsilon > 0$, in order to estimate $|f(x) - f(a)|$, consider two cases: (i) $x < \varepsilon^2$ and $a < \varepsilon^2$ and (ii) $x \geq \varepsilon^2$ or $a \geq \varepsilon^2$, and use multiplication by the conjugate in case (ii).

Problem 2: Let $f(x) = \frac{1}{x}$. Prove that f is NOT uniformly continuous on $(0, 1)$, again directly from the definition.

Problem 3: Let E be a subset of $\mathbb{R}$, and assume that functions $f, g : E \to \mathbb{R}$ are both uniformly continuous on E.

(a) Prove that $f + g$ is uniformly continuous on E
(b) Assume that f and g are bounded on E. Prove that fg is uniformly continuous on E.
(c) Give an example showing that in general fg may not be uniformly continuous on E (if we do not assume that f and g are bounded)
(d) (bonus) Give an example where f is bounded, g is unbounded and fg is not uniformly continuous on E (of course, such an example would also work for (c), but there is a much easier example where both functions are unbounded).

Hint: (a) and (b) can be proved similarly to sum and product rules for limits of sequences.

Problem 4: This problem outlines an alternative proof of the Intermediate Value Theorem which uses uniform continuity. Let $f : [a, b] \to \mathbb{R}$ be a continuous function s.t. $f(a) < 0$ and $f(b) > 0$. We want to prove that there exists $c \in (a,b)$ s.t. $f(c) = 0$. Assume, by way of contradiction, that there is no such c, that is, $f(x) \neq 0$ for all $x \in I$.

(i) Define the function $g : [a,b] \to \mathbb{R}$ by $g(x) = 1$ if $f(x) > 0$ and $g(x) = -1$ if $f(x) < 0$ (note that g is well-defined because we assume that $f(x) \neq 0$ for all $x \in [a,b]$). Prove that g is continuous on $[a,b]$ and hence also uniformly continuous by Theorem 16.1 from class (continuity on closed bounded intervals implies uniform continuity). **Hint:** use the sign preservation lemma (applied to f)

(ii) For each $n \in \mathbb{N}$ define the real numbers $x_{n,0}, x_{n,1}, \ldots, x_{n,n}$ by $x_{i,n} = a + \frac{i}{n}(b-a)$ (geometrically we divide $[a,b]$ into n subintervals of the same length and let $\{x_{i,n}\}_{i=0}^{\infty}$ denote the endpoints of those intervals). Prove that for each n there exists $0 \leq k < n$ s.t. $g(x_{k,n}) = -1$ and $g(x_{k+1,n}) = 1$.

(iii) Now use (ii) to prove that g cannot be uniformly continuous, reaching a contradiction with (a).

Problem 5: Let f be a real function which is uniformly continuous on some set E. Prove that if $\{x_n\}$ is any convergent sequence s.t. $x_n \in E$ for all n, then the sequence $\{f(x_n)\}$ is also convergent. **Note:** If $\{x_n\}$ converges to some $x \in E$, then by the sequential characterization of continuity $\{f(x_n)\}$ converges to $f(x)$ (so uniform continuity is not needed); however, in this problem we claim that $\{f(x_n)\}$ converges regardless of whether $\lim_{n\to\infty} x_n$ lies in E or not. **Hint:** The result of this problem follows immediately from one of the results in Section 3.4 of the book and another theorem we proved earlier in the course.

Problem 6: (bonus). Let $f : \mathbb{Q} \to \mathbb{R}$ (that is, the domain of f is the set of all rational numbers), and suppose that f is uniformly continuous on $\mathbb{Q} \cap [a,b]$ for every closed bounded interval $[a,b]$. The goal of this problem is to show that f can be extended to a continuous function F defined on $\mathbb{R}$.

(a) Let $\{x_n\}$ be a convergent sequence of rational numbers. Prove that $\lim_{n\to\infty} f(x_n)$ exists (this follows easily from Problem 5).

(b) Now let $\{x_n\}$ and $\{y_n\}$ be convergent sequences of rational numbers, and assume that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$. Prove that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$. **Hint:** Assume that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, and use $\{x_n\}$ and $\{y_n\}$ to construct another sequence $\{z_n\}$ s.t. $\{z_n\}$ converges to x, but $\{f(z_n)\}$ does not converge (to any number), reaching a contradiction with (a).

(c) Now define the function $F : \mathbb{R} \to \mathbb{R}$ as follows: given $x \in \mathbb{R}$, choose a sequence $\{x_n\}$ of rational numbers which converges to x, and define

$F(x) = \lim_{n\to\infty} f(x_n)$ (note that the limit on the right-hand side does not depend on the choice of the sequence $\{x_n\}$ by (b)). Prove that $F(x) = f(x)$ if $x \in \mathbb{Q}$ (so that F is indeed an extension of f) and F is continuous (at every point of $\mathbb{R}$). **Hint:** To prove that F is continuous, it suffices to show that F is uniformly continuous on $[a, b]$ for every closed bounded interval $[a, b]$. To prove the latter, show that if we fix $\varepsilon > 0$ and let $\delta > 0$ be such that $|f(x)-f(y)| < \frac{\varepsilon}{2}$ for all $x, y \in [a, b] \cap \mathbb{Q}$ satisfying $|x - y| < \delta$, then $|F(x) - F(y)| < \varepsilon$ holds for all $x, y \in [a, b]$ satisfying $|x - y| < \delta$.

Note: One can use Problem 6, for instance, to rigorously define the function $F(x) = \alpha^x$ for $x \in \mathbb{R}$ (where α is a fixed positive real number).