Homework #7. Due Thursday, March 19th, in class

Reading:

1. For this assignment: 3.3 + class notes (Lectures 14-15). Lecture 15 (snow day) notes will be posted during the break week.

2. For next week's classes: 3.4 (uniform continuity) and 4.1 (the derivative).

Problems:

Problem 1: Prove the SUM RULE part of Theorem 14.3 from class (arithmetic operations preserve continuity = Theorem 3.22 from the book) using sequential characterization of continuity (Theorem 14.2 from class) and arithmetic properties of limits of sequences (Theorem 7.4 from class = Theorem 2.12 from the book). **Note:** Your proof should not involve any epsilons or deltas.

Problem 2: Let $a > 0$ be a real number and n some natural number. Prove that there exists unique positive real number x such that $x^n = a$ (by definition we denote this number x by $\sqrt[n]{a}$). **Hint:** To prove existence of such x apply IVT (intermediate value theorem) to the function $f(x) = x^n - a$ on a suitable closed interval. Consider the cases $a > 1$ and $a < 1$ separately (of course if $a = 1$, we can simply set $x = 1$). To prove uniqueness use the fact that if $0 < x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$ (we proved this in Lecture 5).

Problem 3: Let $f : [a, b] \to [a, b]$ be a continuous function (that is, both domain and codomain of f are the same closed bounded interval $[a, b]$). Prove that f has a fixed point, that is, there exists $c \in [a, b]$ such that $f(c) = c$.

Problem 4: Read the definition of limits at infinity (see § 3.2) before doing this problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist (as finite limits).

(a) Prove that f is bounded (on $\mathbb{R}$). **Hint:** Divide $\mathbb{R}$ into three subintervals I_1, I_2, I_3 (with I_1 being the left interval, I_2 the middle interval and I_3 the right interval) and show that f is bounded on each I_k. To prove that f is bounded on I_2 use Extreme Value Theorem, and to prove that f is bounded on I_1 and I_3 use the fact $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist (for this part you can imitate the proof of Theorem 7.3 which asserts that any convergent sequence is bounded).

(b) Show by an explicit example that f may attain neither minimum nor maximum value on $\mathbb{R}$.

Problem 5: Review the definition of one-sided limits (see § 3.2) before doing this problem. Let I be an open (not necessarily bounded) interval, that is, $I = (c, d)$ where d is a real number or symbol $+\infty$ and c is a real number or symbol $-\infty$.

Let $f : I \to \mathbb{R}$ be a function. We say that f is *increasing* if for all $x, y \in I$, the inequality $x < y$ implies $f(x) \leq f(y)$. The goal of this problem is to show that increasing functions can only have jump discontinuities. By definition, a function f has a *jump discontinuity* at a point a if both one-sided limits $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist (as finite limits) but do not equal each other. In all parts of the problem we fix $a \in I$.

(a) Let $S = \{f(x) : x < a\} = f((c, a))$ and $T = \{f(x) : x > a\} = f((a, d))$. Prove that S is bounded above and T is bounded below and give an explicit bound in both cases.

(b) Prove that $\lim_{x \to a^-} f(x)$ exists and equals $\sup(S)$ and $\lim_{x \to a^+} f(x)$ exists and equals $\inf(T)$. **Hint:** this can be proved similarly to the monotone convergence theorem.

(c) Now use (a) and (b) to prove that either f is continuous at a or f has jump discontinuity at a.

(d) (bonus) Prove that the set of points where f is discontinuous is countable. **Hint:** Construct an injective map from the set of discontinuities of f to $\mathbb{Q}$.

Problem 6: Let D be the Dirichlet function (Example 3.32 in the book) and MD the modified Dirichlet function (Example 3.33 in the book). As proved in § 3.3, D is discontinuous at every point of the real line and MD is discontinuous at all rationals and continuous at all irrationals.

(i) Prove that D has type 2 discontinuity at every point $a \in \mathbb{R}$ (by definition this means that at least one of the one-sided limits at a does not exist).

(ii) Prove that MD has a removable discontinuity at every point $a \in \mathbb{Q}$.